Master's Thesis

Calibration of Ineffective Theorems of Analysis in a Constructive Context

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The existence theorem is not the valuable thing, but the construction carried out in the proof. Mathematics is, as Brouwer sometimes says, more activity than theory.
— Herman Weyl

Valuable distinctions deserve to be maintained.
— Errett Bishop

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Abstract

The main objective of this thesis is to investigate a hierarchy of logical principles not part of intuitionistic logic as developed by Akama et al. (2004). Intuitionistic logic is a basis for constructive reasoning and the levels in the hierarchy correspond to different degrees of non-constructivity.

One of the motivations for studying this hierarchy is a method suggested by Hayashi et al. to test the formalisation of proofs. The domain of this method is called "limit computable mathematics" and is characterised by the hierarchy.

We shall show how a part of the hierarchy has equivalents in well-known theorems of mathematical analysis, such as the intermediate value theorem and the Bolzano-Weierstraß theorem.

A part of proof theory is concerned with the extraction of computational information from classical proofs. This provides semi-constructive interpretations of the hierarchy and our work therefore results in precise calibration of the general ineffectiveness of a series of theorems from mathematical practice.

Resumé

Dette speciales hovedformål er at undersøge et hierarki af ikke-intuitionistiske, logiske principper, der er udviklet af Akama et al. (2004). Intuitionistisk logik er et grundlag for konstruktiv bevisførelse, og hierakiets niveauer svarer til forskellige grader af ikke-konstruktivitet.

Én af motivationerne for at studere dette hierarki er en metode til at teste formaliseringen af beviser, som er foreslået af Hayashi et al. Domænet for denne metode kaldes "limit computable mathematics" og er karakteriseret i hierarkiet.

Vi vil vise, hvordan en del af hierakiet, sat i en konstruktiv kontekst, har ækvivalenter i velkendte sætninger fra matematisk analyse, som f.eks. skæringssætningen og Bolzano-Weierstraß-sætningen.

En del af matematisk bevisteori forsøger at uddrage konstruktiv information fra klassiske beviser. Dette giver en semi-konstruktiv fortolkning af hierarkiet, og resultaterne i dette speciale leverer således præcise kalibreringer af ikke-konstruktiviteten i en serie af konkrete matematiske sætninger.

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Chapter 1

Introduction

The aim of this thesis is to do a constructively based investigation of non-constructivity in mathematical practice.

On the logical level, the difference between constructive and non-constructive reasoning can in a suitable context be boiled down to one axiom; the infamous Law of the Excluded Middle (LEM), $A \vee \neg A$. By characterising proofs in terms of the complexity of A in this axiom, we get a hierarchy that can be used to "calibrate"[1] the non-constructive strength of mathematical theorems.

There are various scales for measuring the "calibre" of a mathematical theorem, each with their own objectives. The one used in this project has some rather new motivations, and therefore its results give, to our knowledge, the first calibrations of their kind.

The main results are also gathered and discussed in [52].

1.1 Constructivism in Mathematics

We give a brief sketch of the history of mathematical constructivism. This is to introduce some of the main characters in — and branches of — constructivism, since we shall make occasional references to them.

Constructivism dates back far in mathematics. For instance, the Danish historian of mathematics H.G. Zeuthen (1839–1920) suggested that the constructions in Euclid's Elements were to be understood as existence proofs.[2] Here we allow ourselves to skip to the beginning of the 20th century where L.E.J. Brouwer (1881–1966) started developing an approach to mathematics

[1] The American Heritage Dictionary of the English Language, Fourth Edition: "To determine the caliber of (a tube)".

[2] However, modern history of mathematics tends to turn from this suggestion.

that later became known as *intuitionism*. It was based on a basically solipsistic philosophy of mathematics that viewed mathematical objects as creations of the individual mind. Gradually Brouwer, almost solitarily, founded an intuitionistic mathematics, and the main critique of classical mathematics may be seen as condensed at two points: The classical interpretation of existence, and the use of logic in general and LEM in particular.

Intuitionism identifies "to exist" with "to be constructed" and by the same line of thought, viewing a disjunction as a binary existence statement, it follows that LEM must be rejected, since one might not be able to decide which of A and $\neg A$ that holds.

A student of Brouwer, A. Heyting (1898–1980), carried out the first formalisation of intuitionistic arithmetic and thereby made intuitionism reach a bigger audience. Though Brouwer's intuitionism was far from captured in Heyting's formal system, the latter, quite surprisingly, *did* manage to express the constructivistic characteristics of the more or less language-less philosophy of intuitionism. Later more features of intuitionism were implemented in the formalisation, for instance the so-called *choice sequences*, which Brouwer used to prove the perplexing statement that "all total functions from $\mathbb{R}$ to $\mathbb{R}$ are continuous". The surprising result stems from the fact that intuitionism is not a restriction of classical mathematics, but an orthogonal mathematical theory.

In the sixties a classical, well-established mathematician E. Bishop (1928–1983) developed a sympathetic new branch of constructive mathematics. He used an intuitionistic logical basis as given by Heyting, and he did not make any non-classical assumptions (such as axioms for choice-sequences). Therefore his results could be read by both classical and intuitionistic mathematicians; and due to the enormous amount of results in his first monograph on the subject [4], this was, and is, indeed an interesting approach.

1.2 Motivations

In general, constructive mathematics has of course a close connection to computer science, since a proof of an existence statement is required to include a method for finding the object that is claimed to exist. For the formal constructive systems that we shall use here, one can verify that constructions *always* give rise to recursive functions. A question of fundamental importance is how this property is affected by allowing restricted use of classical logic. And a second question like it is what amount of classical mathematics that can be carried out under the restrictions.

In the next paragraphs we give some specific motivations for answering the latter question

Learning Theory and Limit Computable Mathematics. One aspect of learning theory deals with a generalised notion of computability where a function f is said to be *limit computable* if there is an effective method that given n outputs a sequence of guesses from some stage on all will be $f(n)$. Limit Computable Mathematics (LCM) is a part of classical mathematics that goes beyond constructive mathematics because it uses methods which are ineffective. On the other hand, the ineffective methods are still, in a certain sense, effective in the limit.

LCM was first motivated by an idea termed Proof Animation to *test* both formal proofs under development and the formalisations of completed proofs. Proof Animation tries to extract constructions (programs) from proofs and then to look for bugs in them. Further research in the area of LCM has characterised it in terms of Heyting's arithmetic, which in turn has proposed a more general investigation of a hierarchy of semi-classical logical principles. In [15], [45] and [1] the research development in this area is described and documented.

This thesis will aim at identifying the levels of the mentioned hierarchy with actual mathematical practice and thereby investigate the scope of LCM.

Reverse Mathematics. The hierarchy from [1] can be seen as an ordering of non-constructive proof methods. As such, another way of viewing the aim of this thesis is as a classification of theorems of classical analysis by their non-constructive strength.

Reverse mathematics is a metamathematical study of a series of formal systems in which increasing parts of mathematics can be carried out. This results in a classification of mathematical theorems, but in terms of axioms of set-theory instead of principles of classical logic.

We shall see that the categorisation made in reverse mathematics is orthogonal to the one we present here, reflecting the phenomenon we mentioned regarding intuitionistic and classical mathematics.

When a large part of mathematics has been fitted in to one of the basic systems of reverse mathematics, insights in mathematical practice can be gained by reflecting on the nature of the basic systems. For instance, weak König's lemma (WKL) — stating that an infinite binary tree has an infinite branch — is seen to play an important role in analysis. Further analysis of WKL reveals that it has interesting connections to recursion theory etc. It turns out that WKL essentially can be expressed in a purely logical form, which gives one of the principles that we shall study here.

Proof Mining. Our classification of theorems is done by logical principles and these principles are told apart by techniques of proof theory which extract the constructive information that the various principles carry. Thereby the mathematical theorems get characterised by the amount of constructive information that they preserve in proofs. A part of proof theory tries to "unwind" concrete proofs, ie.

> determine the constructive (recursive) content [...] of the non-constructive concepts and theorems used in mathematics.
> (G. Kreisel, [36, p. 155])[3]

This part is called *proof mining* and was for semi-classical contexts introduced in [30]. The classification we give in this thesis is of interest in proof mining, since the technique used to separate the logical principles classifies them by the amount of constructive content that one at least can unwind from a proof in which the principle is used.

Constructivism and mathematics. It is a well-known fact that many theorems of classical mathematics are inherently non-constructive. A general motivation of constructive calibration is to measure *how* non-constructive mathematical analysis is, and to isolate, in logical principles, the ineffective modes of reasoning that are used in specific analytic proofs.

This will provide insights in mathematical practice. We shall see both that it characterises mathematical non-constructivity in terms of natural axiom instances from classical logic, and that it also points to logical principles that apparently do not occur in a large part of mathematics as necessary axioms.

1.3 Thesis Structure

The report falls in three parts. The first part defines the context in which the calibrations are to be performed and the logical, semi-classical principles that are to be used. This also includes consideration of other possible contexts with a justification for the actual choice that has been made. The first part ends with Chap. 3 which introduces a number of so-called proof interpretations that will be used for separating the logical principles.

The second part establishes the lower levels of the hierarchy and discusses how to interpret the various logical principles. A small survey of some of the related work then follows in Chap. 5.

The last three chapters are devoted to the actual calibrations, which naturally fall in three pairs.

[3]Original emphasis has been removed.

Chapter 2

Setting up the Context

Most of the motivations mentioned in the introduction — here we have reverse mathematics, proof mining and constructivism in mind — leave important choices regarding the context open to us. Systems resulting from many of these choices have been studied previously; cf. Chap. 5 for an overview of the work of relevance to this project. This chapter first give a discussion of the chosen context and its motivation, and it will then describe the formal context in some detail.

2.1 Motivating the Context

To motivate the context we first need to give a suitable characterisation of limit computable mathematics.

2.1.1 Limit Computable Mathematics

It is known that proofs using only constructive means are in the scope of proof animation. This seems reasonable by the intuitive content of "constructive". But it turns out that even for a more general class of proofs, it is possibly to extract programs that can be used for proof animation. Limit Computable Mathematics is such a class.

We say that f is the limit of the total recursive function g if $\exists t_0 \forall t \geq t_0 f(\underline{x}) = g(t, \underline{x})$. Stated in terms of the arithmetical hierarchy, a set is Δ^0_2 iff its characteristic function is the limit of a recursive function (cf. [48, 5.4]).

In [45], LCM is characterised by the formal theory which is obtained by adding the classical logical axiom $\neg\neg\exists x \forall y A_0(x, y) \to \exists x \forall y A_0(x, y)$ (A_0 quantifier-free) to the intuitionistic counterpart of Peano Arithmetic, which is called Heyting Arithmetic HA. We denote this classical principle Σ^0_2-DNE

(Double Negation Elimination); which too is the name we shall use when these notions get defined formally. Σ^0_2-DNE is, as we shall see later, non-constructive, but it is easy to see why it is limit-constructive: The task is to come up with a function that, in the parameters, finds the x claimed to exist in the conclusion of Σ^0_2-DNE. We simply try to let $x = 0$ and go through the y's. If we find a y such that $\neg A_0(x, y)$ (which at each step can be checked, since A_0 is quantifier-free; cf. Sect. 2.2.2), we change our mind and instead try $x = 1$, etc. This procedure is limit computable, since by the premise of Σ^0_2-DNE we are guaranteed that at some stage we find an x such that $\forall y A_0(x, y)$. However, this is (in general) not computable, for we might not recognise the correct x.

In learning theoretic terms, the function realizing Σ^0_2-DNE can be *learned*, since only finitely many mind-changes are needed to find the correct answer. It also corresponds to a very general type of learning, since we cannot even compute a bound on how many revisions are needed.[1]

2.1.2 Context Choices

Based on the discussion above, we now describe some of the choices that have resulted in the context we introduce in the following sections.

The first decision to make is on which logical, respectively non-logical, axioms and rules that are to be a part of the context. Apart from classical logic, one might be interested in one of the refinements, linear logic and intuitionistic logic. Linear logic is a resource sensitive approach, which keeps track of how many times a hypothesis is used in the derivation of some conclusion. This is, roughly speaking, achieved by disallowing general contractions and weakenings.[2] Especially in computer science, linear logic has found applications.

Both classical and linear logic may give rise to contexts that are also of interest for proof theory and constructivity or computer science. However, for the purpose of LCM, which has an intuitionistic core, it is far more natural to work in an intuitionistic context.

As to the non-logical axioms, one might consider a system based on classical logic, but with only weak set-existence axioms, such as comprehension and the axiom of choice, and restricting the available induction to thereby have it be constructive in some sense. But on an intuitionistic basis, the axiom of choice, at least its countable restriction, is harmless since it only

[1] [1] has a discussion of the various kinds of learning and their relation to some of the semi-classical principles that will be examined in the later chapters.

[2] Contraction is expressed by the axioms $A \to A \wedge A$ and $A \vee A \to A$. For weakening, cf. the axiom list on p. 12.

makes the constructive interpretation of the quantifiers explicit. Still, we will be explicit when using the axiom of choice. Restricted induction is useful for the study of sub-recursive fragments of both classical and intuitionistic systems, but again, since induction is unproblematic in LCM, we allow full induction.

The next issue we discuss is what we are going to consider an instance of an axiom schema or mathematical principle. For an informal illustration, consider the Σ^0_1-LEM schema,

$$\exists x(t(x,y)=0) \vee \neg\exists x(t(x,y)=0) \ .$$

It is enough to consider matrices[3] of this simple kind since the systems we use reduce quantifier-free formulas to equality test between terms. And as usual, blocks of quantifiers can be coded together such that it is only necessary to consider formulas with alternating quantifiers.

In a higher-order[4] setting with the axiom of choice, we could from Σ^0_1-LEM prove the existence of a function f such that

$$\forall y\exists x(t(x,y)=0) \leftrightarrow \forall y(t(f(y),y)=0) \ ;$$

that is, f absorbs an $\exists$-quantifier. This may be iterated to get $A \vee \neg A$ for any arithmetical A, hence full classical logic for any arithmetical formulas. To precisely calibrate the scope of LCM, we need somehow block the possibility of such iteration, since, as will be clear later, Σ^0_2-DNE implies Σ^0_1-LEM and is, in a certain sense, the strongest principle that has a limit computable realizer. We do this by only allowing instances to be given by terms with parameters of number-type.

2.2 Formal Context

The purpose of this section is to set up the formal context in which this project works.

2.2.1 Intuitionistic Predicate Logic

We begin by describing intuitionistic predicate logic (IL) — our aim is not to give an introduction to predicate logic as such. We shall therefore be brief

[3]The matrix is the formula part following the explicit quantifiers. We shall not need a more formal definition.

[4]The higher-order context that we will use is conservative over HA, [54, 3.6.2].

and adopt the usual conventions concerning notation, naming of variables, omission of parenthesis etc. Details on this can be found in [2] and [54, 1.1].

As logical connective we have $\wedge, \vee, \exists, \forall, \rightarrow, \perp$.[5] We use the abbreviations $\neg A :\equiv A \rightarrow \perp$ and $A \leftrightarrow B :\equiv (A \rightarrow B) \wedge (B \rightarrow A)$.

Recall that terms are expressions built up from variables, constants and function symbols; atomic formulas are built up from terms and equality and relation symbols of the language. For a definition of "formulas", "free variables" etc. we again refer to [2].

The proof system we use is a so-called Hilbert-type system (ie. it is based on axioms and inference rules, as opposed to, for instance, "sequent calculi" and "natural deduction systems") suggested by Spector as given in [54, 1.1.3].

Axioms of IL.

(i) $A \rightarrow A$.

(ii) $A \wedge B \rightarrow A\,,\ A \wedge B \rightarrow B$.

(iii) $A \rightarrow A \vee B\,,\ B \rightarrow A \vee B$ (weakening).

(iv) $\perp \rightarrow A$ (*ex falso quodlibet*).

(v) $\forall x A(x) \rightarrow A(t)$, if t is substitutable for x in A.[6]

(vi) $A(t) \rightarrow \exists x A(x)$, if t is substitutable for x in A.

Inference Rules of IL.

(i)

$$\frac{A \quad A \rightarrow B}{B} \quad (\textit{modus ponens}) \ .$$

(ii)

$$\frac{A \rightarrow B \quad B \rightarrow C}{A \rightarrow C} \quad (\textit{syllogism}) \ .$$

(iii)

$$\frac{A \rightarrow C \quad B \rightarrow C}{A \vee B \rightarrow C}\,,\ \frac{A \rightarrow B \quad A \rightarrow C}{A \rightarrow B \wedge C} \ .$$

(iv)

$$\frac{A \wedge B \rightarrow C}{A \rightarrow (B \rightarrow C)}\,,\ \frac{A \rightarrow (B \rightarrow C)}{A \wedge B \rightarrow C} \ .$$

[5] Pronounced "and", "or", "there exists", "for all", "implies", "absurdity".

[6] That is, no variable in t becomes bound after the substitution.

(v)

$$\frac{B \to A(x)}{B \to \forall x A(x)}\,, \quad \frac{A(x) \to B}{\exists x A(x) \to B} \quad \text{, where } x \text{ is not free in } B.$$

We write $\mathsf{IL} \vdash$ for deducibility in the calculus above. If we add further axioms Γ to the calculus, deducibility is then denoted $\mathsf{IL} + \Gamma \vdash$. As in most other reasonable proof systems, the deduction theorem is valid in the Hilbert style calculus we use.

Theorem 2.1. *For any set of axioms* Γ *and any closed formula* A*, we have*

$$\mathsf{IL} + \Gamma \cup \{A\} \vdash B \Rightarrow \mathsf{IL} + \Gamma \vdash A \to B \ .$$

Example 2.2. As an example of a derivation we show the following, $B \to (\neg B \to C)$.

$$\begin{prooftree}
\AxiomC{$\bot \overset{\text{(iv)}}{\to} C$}
\AxiomC{$B \wedge \neg B \overset{\text{(i)}}{\to} B \wedge \neg B$}
\AxiomC{$B \wedge \neg B \overset{\text{(ii)}}{\to} B$}
\LeftLabel{(iii)}
\BinaryInfC{$B \wedge \neg B \to [B \wedge \neg B] \wedge B$}
\AxiomC{$B \wedge \neg B \overset{\text{(ii)}}{\to} (B \to \bot)$}
\RightLabel{(iv)}
\UnaryInfC{$[B \wedge \neg B] \wedge B \to \bot$}
\LeftLabel{(ii)}
\BinaryInfC{$B \wedge \neg B \to \bot$}
\RightLabel{(ii)}
\BinaryInfC{$B \wedge \neg B \to C$}
\LeftLabel{(iv)}
\UnaryInfC{$B \to (\neg B \to C)$}
\end{prooftree}$$

Note that this proof tree incorporates a deduction of $\neg(B \wedge \neg B)$.

Once we have the deduction theorem, $\mathsf{IL} \vdash B \to (\neg B \to C)$ can be derived more intuitively: By the deduction theorem (applied twice) it is enough to show $\mathsf{IL} + \{B, \neg B\} \vdash C$.

$$\begin{prooftree}
\AxiomC{B}
\AxiomC{$\neg B$}
\LeftLabel{(i)}
\BinaryInfC{$\bot$}
\AxiomC{$\bot \overset{\text{(iv)}}{\to} C$}
\RightLabel{(i)}
\BinaryInfC{C}
\end{prooftree}$$

For the historically interested, the just derived formula is the main difference between Heyting's formalisation of intuitionistic logic and A.N. Kolmogorov's (1903–1987) preceeding and partial ditto. Kolmogorov argues that the principle "does not have and cannot have any intuitive foundation since it asserts something about the consequences of something impossible: we have to accept [C] if the true judgement [B] is regarded as false"[7].

[7][35, II §4].

Basic Properties. We give a list of some theorems that are all derivable in IL to give the reader some impression of what can and what cannot be done in an intuitionistic setting.

1. $A \to \neg\neg A$
2. $\neg A \leftrightarrow \neg\neg\neg A$
3. $\neg(A \vee B) \leftrightarrow \neg A \wedge \neg B$
4. $\neg A \vee \neg B \to \neg(A \wedge B)$
5. $(\neg\neg A \wedge \neg\neg B) \leftrightarrow \neg\neg(A \wedge B)$
6. $(A \to B) \to (\neg B \to \neg A)$
7. $(\neg\neg A \to \neg\neg B) \leftrightarrow \neg\neg(A \to B)$ and $\neg\neg(A \to B) \leftrightarrow (A \to \neg\neg B)$
8. $\neg\exists x A(x) \leftrightarrow \forall x \neg A(x)$
9. $\exists x \neg A(x) \to \neg\forall x A(x)$
10. $\neg\neg\forall x A(x) \to \forall x \neg\neg A(x)$
11. $\forall x(A(x) \to B) \leftrightarrow (\exists x A(x) \to B)$
12. $\exists x(A(x) \to B) \to (\forall x A(x) \to B)$
13. $\forall x(B \to A(x)) \leftrightarrow (B \to \forall x A(x))$
14. $\exists x(B \to A(x)) \to (B \to \exists x A(x))$

Proofs and an extended list can be found in [8, 5.2].

The Excluded Axiom. Adding the axiom schema

(vii) $A \vee \neg A$ (*tertium non datur*),

to the list of axioms results in a formalisation of classical logic. Thus, we refind the curious fact that the difference between constructive and classical reasoning can be isolated in exactly one axiom (if we for the moment believe that intuitionistic logic corresponds to constructive reasoning).

We shall use the acronym LEM for the axiom, referring to (one of) its English name(s) — Law of the Excluded Middle. This, in turn, refers to the interpretation of LEM that for a formula A there are only two possibilities, either A is valid or else $\neg A$ is valid; there is nothing between the two. Note that from 3. in the list above and Example 2.2, one finds that $\mathsf{IL} \vdash \neg\neg(A \vee \neg A)$.

Alternatively we could have added the schema (vii′) $\neg\neg A \to A$.

This would again give a formalisation of classical logic. We shall denote this schema DNE, which is an acronym for Double Negation Elimination.

2.2.2 Heyting Arithmetic

This section describes the intuitionistic first-order theory of arithmetic, Heyting arithmetic (HA), which is the intuitionistic counterpart of Peano arithmetic (PA); ie. HA and PA have the same language and non-logical axioms. We denote primitive recursive arithmetic, which is PA with only quantifier-free induction, with PRA.

The language of HA contains the constant symbol 0, a unary function symbol S (the successor), function symbols for all primitive recursive functions and equality $=$ as a binary predicate.

Non-Logical Axioms of HA.

Equality axioms. $x = x, x = y \wedge z = y \to x = z$. And for every n-ary function constant f, $\underline{x} = \underline{y} \to f(\underline{x}) = f(\underline{y})$.[8]

Successor axioms. $S(x) \neq 0, S(x) = S(y) \to x = y$.

Defining axioms for the primitive recursive functions.

Induction axiom (IA). The induction schema[9]

$$A(0) \wedge \forall x(A(x) \to A(S(x))) \to \forall x A(x) \ .$$

The usual pairing and coding of finite sequences can be carried out in HA and we use $\langle m, n\rangle$ to denote the code of the tuple consisting of m and n. We shall use $\mathrm{sg}(\cdot)$ for the sign (or *signum*) function ($\mathrm{sg}(0) = 0$, $\mathrm{sg}(n+1) = 1$) and $\overline{\mathrm{sg}}$ for its complement ($\overline{\mathrm{sg}}(0) = 1$, $\overline{\mathrm{sg}}(n+1) = 0$).

Any quantifier-free formula is equivalent to an atomic formula and they are decidable; ie. the quantifier-free instances of LEM are provable in HA.

[8] We underline variables to denote finite tuples, $\underline{x} \equiv x_1, \ldots, x_n$. Equality between tuples are taken componentwise.

[9] The schematic letter A stands for any formula of the language of HA. Ie. restrictions as the one mentioned in Sect. 2.1 are not made here.

2.2.3 Heyting Arithmetic in All Finite Types

This section extends HA to an intuitionistic theory dealing with objects of higher types. Among several established systems doing this we have chosen HA^ω from [54, 1.6.15]. It has mainly been studied in relation with Gödel's functional interpretation (cf. Sect. 3.3), but has the, for our purpose, useful properties that it is the weakest system[10] and can be used throughout all of this project.

We shall use PA^ω for the classical variant of HA^ω.

The Type Structure T. The set of finite types T is defined inductively by the rules

(i) $0 \in \mathsf{T}$.

(ii) If $\sigma, \tau \in \mathsf{T}$, then $(\sigma \to \tau) \in \mathsf{T}$.

0 is the type of the natural numbers. Functions from type σ to τ are of type $\sigma \to \tau$. The usual convention that "$\to$" between types binds to the right is adopted here, such that $\rho \to \sigma \to \tau$ is $\rho \to (\sigma \to \tau)$. We assign natural numbers to the so-called *pure types*; 1 denotes $(0 \to 0)$ and $n+1$ denotes $(n \to 0)$.

We use $\deg(\tau)$ to denote the degree of a type τ:

$$\begin{aligned} \deg(0) &= 0 \ , \\ \deg(\sigma \to \tau) &= \max\{\deg(\sigma)+1, \deg(\tau)\} \ . \end{aligned}$$

Language of HA^ω. HA^ω is based on many-sorted[11] intuitionistic predicate logic. That is, for each type τ (sort) there are variables $x^\tau, y^\tau, z^\tau, \ldots$. The language of HA^ω contains the constant 0^0 (zero, of type 0), S^1 (successor of type 1), $\Pi_{\sigma,\tau}$ (projector of type $\sigma \to \tau \to \sigma$), $\Sigma_{\rho,\sigma,\tau}$ (combinator of type $(\rho \to \sigma \to \tau) \to (\rho \to \sigma) \to \rho \to \tau$) and R_τ (recursor of type $0 \to \tau \to (\tau \to 0 \to \tau) \to \tau$). Finally there is a binary equality predicate $=_0$ between objects of type 0.

The terms of HA^ω are defined by two clauses:

1. Constants and variables are terms
2. If s is a term of type σ and t is a terms of type $\sigma \to \tau$, then $t(s)$ is a term of type τ.

Instead of $t(s)(r)$ we shall simply write $t(s, r)$ etc.

[10] It is contained in both $\mathsf{N\text{-}HA}^\omega$ and $\mathsf{WE\text{-}HA}^\omega$ from [54].

[11] Cf. [2, 5.1].

Non-logical Axioms of HA^ω.

Equality axioms for $=_0$.

Successor axioms as above.

Induction schema as above (with instances to be taken from the new language).

Substitutivity schemata. For type 0 objects,

$$x^0 = y^0 \to t[x^0] = t[y^0] \ ,$$

and the following for terms t of type 0,

$$\begin{aligned} t[\Pi(x,y)] &= t[x] \\ t[\Sigma(x,y,z)] &= t[x(z,y(z))] \\ t[R(0,y,z)] &= t[y] \\ t[R(S(x),y,z)] &= t[z(R(x,y,z),x)] \ . \end{aligned}$$

Useful properties of HA^ω. The λ operator is definable in HA^ω and substitutivity gives for t of type 0: $t[(\lambda x^\tau.t'[x])(t'')] = t[t'[t'']]$.

HA is a a subsystem of HA^ω in the sense of [54, 1.6.9]. And in this sense, HA^ω is conservative over HA, [54, 3.6.2].

Models of HA^ω HA^ω does not commit itself to either of the two interpretations of equality in higher types; *intensional* and *extensional* equality. The intensional interpretation is that two objects of higher types are equal if they are given by the same "rule", and equality between "rules" is thought to be decidable. In contrast, objects are said to be extensionally equal if they give the same values on equal arguments.

One model reflecting intensional equality is HRO (Hereditarily Recursive Operations), which at type 1 consists of the total recursive functions with equality taken to be equality of the codes. A model in the same spirit but reflecting extensional equality is HEO (Hereditarily Effective Operations), which at type 1 again are the total recursive functions, but two functions are equal if they "output" the same natural numbers on equal number "inputs".

Definitions of HRO and HEO are given in [54, 2.4].

The weakness of HA^ω is reflected by the fact that both HRO and HEO are models of HA^ω. This calls for some caution when working with HA^ω as the basic proof system since very natural modes of reasoning based on the extensional interpretation of equality cannot be carried out in HA^ω. This has practical relevance since proofs in HA^ω hardly ever are written as formal proof trees, but instead at a convincing level of detail that show how a formal proof would go. That is, there are some gaps in which one could suspect uses of extensionality to reveal itself under formalisation. However, since results in this thesis at most involve types of degree ≤ 1, extensionality turns out not to be a problematic issue:

By applying elimination of extensionality from [40] we get that adding extensionality to HA^ω is conservative for formulas with all variables of degree ≤ 1, since functions of type 1 are provably extensional in HA^ω.

2.2.4 Higher Type Relations

Definition 2.3. For any type τ we inductively define $x \geq_\tau y$ between objects of type τ.

$$\begin{aligned} x \geq_0 y &:\equiv x \geq y \ , \\ x \geq_{\sigma\to\tau} y &:\equiv \forall z^\sigma \, (x(z) \geq_\tau y(z)) \ . \end{aligned}$$

Definition 2.4. For any type τ we define a relation $x \operatorname{maj}_\tau y$ (x majorizes y) between objects of type τ inductively on τ.

$$\begin{aligned} x \operatorname{maj}_0 y &:\equiv x \geq y \ , \\ x \operatorname{maj}_{\sigma\to\tau} y &:\equiv \forall z_1^\sigma, z_2^\sigma \, (z_1 \operatorname{maj}_\sigma z_2 \to x(z_1) \operatorname{maj}_\tau y(z_2)) \ . \end{aligned}$$

Theorem 2.5 (W. A. Howard [19]). *For any closed term t^ρ of HA^ω, there exists a closed term t'^ρ such that*

$$\mathsf{HA}^\omega \vdash t' \operatorname{maj}_\rho t \ .$$

2.3 Logical and Set-Theoretical Principles

This section defines the logical principles that will be used for the actual calibrations of the final part of the thesis. Also a few set-theoretical principles are given, in particular the axiom of choice.

Definition 2.6. We say that a formula A of HA^ω is $\exists$-free if it is built up from atomic formulas by only $\wedge$, $\to$ and $\forall$; that is, A neither contains $\exists$ nor $\vee$.

Definition 2.7. Let t be a term of HA representing a type 1 function and n a natural number. We define the following two formulas with alternating quantifiers

(i) $\Sigma^0_n(t) :\equiv \exists k_1 \forall k_2 \ldots \mathrm{Q} k_n (t(k_1, k_2, \ldots, k_n) = 0)$, where Q is $\exists$ if n is odd, $\forall$ if n is even.

(ii) $\Pi^0_n(t) :\equiv \forall k_1 \exists k_2 \ldots \mathrm{Q} k_n (t(k_1, k_2, \ldots, k_n) = 0)$, where Q is $\forall$ if n is odd, $\exists$ if n is even.

By Σ^0_n (Π^0_n) we refer to the class of all $\Sigma^0_n(t)$ ($\Pi^0_n(t)$) instances with t being a term of HA. We shall informally use Δ^0_1 to denote the class of formulas that (provably) can be written both as a Σ^0_1 and a Π^0_1 formula.

Having defined the two fundamental classes of formulas above, we proceed by restricting classical logical principles to these classes and thereby obtaining three important series of schemata. An instance of some schema S is given by a term t in HA, possibly with number parameters. In this way $\forall x \mathsf{S}(t[x])$ is an instance of S. However, function parameters are not allowed.

Definition 2.8. The Law of the Excluded Middle principle for Σ^0_n respectively Π^0_n formulas is given by:

$$\Gamma^0_n\text{-}\mathsf{LEM}(t) :\equiv \Gamma^0_n(t) \vee \neg \Gamma^0_n(t) \ ,$$

where Γ is one of Σ and Π.

The more vivid name 'principle of omniscience' was introduced in [4] for the full LEM principle, and 'limited principle of omniscience' (LPO) for a principle stating:

> If $\{a_n\}$ is any decision sequence [ie. a nondecreasing sequence of 0's and 1's], then either all $a_n = 0$ or some $a_n = 1$.

Furthermore the term 'weak limited principle of omniscience' (WLPO) is used in [41] for the principle:

> If $\{a_n\}$ is any decision sequence, then either all $a_n = 0$ or it is contradictory that some $a_n = 1$.

The restriction to decision sequences is inessential, and one therefore notices the resemblance to the schemata Σ^0_1-LEM and Π^0_1-LEM. The difference is that the omniscience principles traditionally are allowed to have function parameters when considered in Bishop's constructive mathematics.

Definition 2.9. The Double Negation Elimination principle for Σ^0_n respectively Π^0_n formulas is given by

$$\Gamma^0_n\text{-}\mathsf{DNE}(t) :\equiv \neg\neg\Gamma^0_n(t) \to \Gamma^0_n(t) \ .$$

Remark 2.10. Π^0_{n+1}-DNE is equivalent to Σ^0_n-DNE in HA and Π^0_1-DNE is provable in HA since $\mathsf{HA} \vdash \neg\neg\forall k A(k) \to \forall k \neg\neg A(k)$, wherefore we shall only consider instances of Σ^0_n-DNE.

For historical reasons Σ^0_1-DNE is also known as Markov's principle, see Sect. 4.1.1. The principle is generalised in higher types by the following.

Definition 2.11. Let

$$\mathsf{M}^\tau \ : \ \neg\neg\exists x^\tau A_0(x) \to \exists x^\tau A_0(x) \ ,$$

for A_0 quantifier-free. Define Markov's principle in all finite types $\mathsf{M}^\omega :\equiv \bigcup_{\tau\in\mathsf{T}}\{\mathsf{M}^\tau\}$.

Definition 2.12. The Lesser Limited Principle of Omniscience for Σ^0_n formulas is given by

$$\Sigma^0_n\text{-}\mathsf{LLPO}(t_1,t_2) :\equiv \neg(\Sigma^0_n(t_1) \wedge \Sigma^0_n(t_2)) \to \Pi^0_n(\overline{\mathrm{sg}}(t_1)) \vee \Pi^0_n(\overline{\mathrm{sg}}(t_2)) \ .$$

The principle is named after the principle LLPO from constructive mathematics. In [20] LLPO reads as follows

> given any [decision sequence $\{a_n\}$] either $a_{2n} = 0$ for all n, or else $a_{2n+1} = 0$ for each n.

The resemblance to Σ^0_1-LLPO is clear. Note also the nondeterministic behaviour — in principle, both $\Pi^0_n(\overline{\mathrm{sg}}(t_1))$ and $\Pi^0_n(\overline{\mathrm{sg}}(t_2))$ might hold, and in such a case the principle, *per se*, will not tell us so and is even free to chose which of the disjuncts to point to.

Definition 2.13. The Independence-of-Premise principle for Σ^0_n respectively Π^0_n premises is given by

$$\Gamma^0_n\text{-}\mathsf{IP}(t) :\equiv (\Gamma^0_n(t) \to \exists l A(l)) \to \exists l(\Gamma^0_n(t) \to A(l)) \ ,$$

where $l \notin \mathrm{FV}(\Gamma^0_n(t))$ and A is an arbitrary formula.

We shall also need two higher-type Independence-of-Premise principles. Let

$$\begin{aligned} \mathsf{IP}^{\sigma,\tau}_\forall \ &: \ (\forall y^\sigma A_0(y) \to \exists x^\tau B(x)) \to \exists x^\tau(\forall y^\sigma A_0(y) \to B(x)) \\ \mathsf{IP}^\tau_{\mathrm{ef}} \ &: \ (A_{\mathrm{ef}} \to \exists x^\tau B(x)) \to \exists x^\tau(A_{\mathrm{ef}} \to B(x)) \ , \end{aligned}$$

for A_0 quantifier-free and A_{ef} $\exists$-free and then define $\mathsf{IP}^\omega_\forall :\equiv \bigcup_{\sigma,\tau\in\mathsf{T}}\{\mathsf{IP}^{\sigma,\tau}_\forall\}$ and $\mathsf{IP}^\omega_{\mathrm{ef}} :\equiv \bigcup_{\tau\in\mathsf{T}}\{\mathsf{IP}^\tau_{\mathrm{ef}}\}$.

Remark 2.14. Note that Γ^0_n-IP follows from Γ^0_n-LEM:
Suppose

$$\Gamma^0_n(t) \to \exists l A(l) \ . \tag{2.1}$$

By Γ^0_n-LEM we get

$$\Gamma^0_n(t) \vee \neg\Gamma^0_n(t) \ .$$

So in the first case by (2.1) also $\exists l A(l)$. For this l we get

$$\Gamma^0_n(t) \to A(l) \ ,$$

hence $\exists l(\Gamma^0_n(t) \to A(l))$. In the other case we get

$$\Gamma^0_n(t) \to A(0) \ ,$$

by $B \to (\neg B \to C)$ (Example 2.2). So in particular we have $\exists l(\Gamma^0_n(t) \to A(l))$ in either case.

Definition 2.15. The Comprehension Axiom schema for Σ^0_n respectively Π^0_n formulas is given by

$$\Gamma^0_n\text{-}\mathsf{CA}(t) :\equiv \exists f^1 \forall l (f(l) = 0 \leftrightarrow \Gamma^0_n(t[l])) \ .$$

The following principle, which can be proved by induction in HA, will become useful.

Definition 2.16. The Collection Principle CP is given by

$$\forall a (\forall x \leq a \exists y A(x,y) \to \exists z \forall x \leq a \exists y \leq z A(x,y)) \ .$$

The Collection Principle can be viewed as a finite version of the Axiom of Choice.

Definition 2.17. The Axiom of Choice $\mathsf{AC}^{\sigma,\tau}$ for type σ, τ is give by

$$\forall x^\sigma \exists y^\tau A(x,y) \to \exists Y^{\sigma\to\tau} \forall x^\sigma A(x, Y(x)) \ .$$

Let $\mathsf{AC} :\equiv \bigcup_{\sigma,\tau\in\mathsf{T}} \{\mathsf{AC}^{\sigma,\tau}\}$.

2.4 Analysis in HA^ω

We shall later study certain theorems of classical analysis. But so far our formal system is still arithmetical. The purpose of the present section is to make it possible to deal with more complex objects, such as rational and real numbers and continuous functions.

2.4.1 Real Numbers

We are going to represent real numbers as Cauchy sequences of rational numbers converging with rate 2^{-n}.

Rational numbers can be represented as pairs of natural numbers in such a way that every natural number, interpreted as the code of a pair, represents a rational number. Following [26] we take $\langle n, m\rangle$ to represent

$$\frac{\frac{n}{2}}{m+1} \quad \text{if } n \text{ is even, and} \qquad\qquad \frac{-\frac{n+1}{2}}{m+1} \quad \text{if } n \text{ is odd.}$$

Furthermore, the basic relations and operations on rationals — $=_{\mathbb{Q}}$,$\geq_{\mathbb{Q}}$, $>_{\mathbb{Q}}$, $+_{\mathbb{Q}}$ and $\cdot_{\mathbb{Q}}$ (multiplication) — can be defined primitive recursively and their basic properties can be verified in HA.

Real numbers are represented as Cauchy sequences of rationals with rate 2^{-n}; hence, by the representation described above, number theoretic functions f^1 satisfying

$$\forall n^0(|f(n) -_{\mathbb{Q}} f(n+1)|_{\mathbb{Q}} <_{\mathbb{Q}} 2^{-n-1}) \ . \tag{2.2}$$

Here it is implicitly understood that "2^{-n-1}" denotes a natural number which is the representation of the actual rational number 2^{-n-1}. Note that the function which, given n, outputs the representation of the rational 2^{-n-1} is easily defined by primitive recursion.

By iterating (2.2), we get the usual (fixed rate) Cauchy criterion,

$$\forall n^0 \forall k^0, m^0 \geq n(|f(m) -_{\mathbb{Q}} f(k)|_{\mathbb{Q}} <_{\mathbb{Q}} 2^{-n}) \ . \tag{2.3}$$

Using a construction from [26, p. 48] we can make every type 1 function represent a real number, ie. quantification over reals is implemented as quantification over type 1 functions: We define in HA^ω a term representing a functional Φ such that

$$\Phi(f) :\equiv \lambda n.f\left(\mu\, k \leq n\left[k = n \vee |f(k) -_{\mathbb{Q}} f(k+1)|_{\mathbb{Q}} \geq_{\mathbb{Q}} 2^{-k-1}\right]\right) \ .$$

Then $\Phi(f)$ is always a Cauchy sequence (in the sense of (2.2)) and if already f satisfied (2.2), then $\forall n(f(n) = (\Phi(f))(n))$.

We write $\widehat{f}$ to denote $\Phi(f)$.

Ordering. Equality on the representation of real numbers is defined by

$$f =_{\mathbb{R}} g :\equiv \forall n \left(|\widehat{f}(n+1) -_{\mathbb{Q}} \widehat{g}(n+1)|_{\mathbb{Q}} <_{\mathbb{Q}} 2^{-n} \right) \ . \tag{2.4}$$

In the same spirit, we have the following definitions.

$$f <_{\mathbb{R}} g :\equiv \exists n (\widehat{g}(n+1) -_{\mathbb{Q}} \widehat{f}(n+1) \geq_{\mathbb{Q}} 2^{-k}) \ ,$$
$$f \leq_{\mathbb{R}} g :\equiv \neg (f >_{\mathbb{R}} g) \ .$$

Note that $=_{\mathbb{R}}$, $<_{\mathbb{R}}$ and $\leq_{\mathbb{R}}$ are not quantifier-free, and in general not decidable. That equality on the representations corresponds exactly to equality between the real numbers represented follows using basic properties of Cauchy sequences from analysis, or alternatively the next lemma; it states that if two representatives have the same limit — in the sense that they get arbitrarily close — then they are equal in the sense of (2.4) (that is, they get close even with rate 2^{-n}).

Lemma 2.18.

$$\mathsf{HA}^{\omega} \vdash f =_{\mathbb{R}} g \leftrightarrow \forall k \exists n \forall m > n (|\widehat{f}(m) -_{\mathbb{Q}} \widehat{g}(m)|_{\mathbb{Q}} <_{\mathbb{Q}} 2^{-k}) \ .$$

Proof. Left to right follows from the properties of the $\widehat{\ }$ construction and (2.2). For the other direction, assume

$$\exists n' (|\widehat{f}(n'+1) -_{\mathbb{Q}} \widehat{g}(n'+1)|_{\mathbb{Q}} \geq_{\mathbb{Q}} 2^{-n'}) \ .$$

Then, by (2.2),

$$\exists n' (|\widehat{f}(n'+2) -_{\mathbb{Q}} \widehat{g}(n'+2)|_{\mathbb{Q}} >_{\mathbb{Q}} 2^{-n'-1}) \ ,$$

and so,

$$\exists n (|\widehat{f}(n+1) -_{\mathbb{Q}} \widehat{g}(n+1)|_{\mathbb{Q}} \geq_{\mathbb{Q}} 2^{-n} + q) \ ,$$

for some rational $q >_{\mathbb{Q}} 0$. By the $\widehat{\ }$ construction and (2.3) we have,

$$\forall m \geq n+1 (|\widehat{f}(m) -_{\mathbb{Q}} \widehat{f}(n+1)|_{\mathbb{Q}} <_{\mathbb{Q}} 2^{-n-1}) \ ,$$

and the same for g. Hence,

$$\begin{aligned} \exists n \forall m \geq n+1 (|\widehat{f}(m) -_{\mathbb{Q}} \widehat{g}(m)|_{\mathbb{Q}} & \\ &=_{\mathbb{Q}} |\widehat{f}(n+1) - \widehat{g}(n+1) - \widehat{g}(m) + \widehat{g}(n+1) + \widehat{f}(m) - \widehat{f}(n+1)| \\ &>_{\mathbb{Q}} 2^{-n} + q - 2^{-n-1} - 2^{-n-1} \\ &=_{\mathbb{Q}} q) \ , \end{aligned}$$

which contradicts the implicative assumption. That is, we have proved $\neg\exists$, which implies $\forall\neg$. Since the prime formulas are decidable in HA, we get the $\widehat{f} =_{\mathbb{R}} \widehat{g}$. $\square$

Basic Properties. The functions $|\cdot|_{\mathbb{R}}$, $+_{\mathbb{R}}$ and $-_{\mathbb{R}}$ may be defined such that

$$\begin{aligned} |f|_{\mathbb{R}}(n) &:= |\widehat{f}(n)|_{\mathbb{Q}} \ , \\ (f +_{\mathbb{R}} g)(n) &:= \widehat{f}(n+1) +_{\mathbb{Q}} \widehat{g}(n+1) \ , \\ (f -_{\mathbb{R}} g)(n) &:= \widehat{f}(n+1) -_{\mathbb{Q}} \widehat{g}(n+1) \ . \end{aligned}$$

One easily gets the useful fact that a (representation of a) real number and its k'th approximation are close by 2^{-k}:

$$\forall k \left(\left| f -_{\mathbb{R}} \widehat{f}(k) \right|_{\mathbb{R}} <_{\mathbb{R}} 2^{-k} \right) \ .$$

Here we have introduced the convention that a rational r in an $\mathbb{R}$-relation is implicitly to be replaced by its trivial $\mathbb{R}$-representation $\lambda n.r$.

The operations and relations defined on representations provably satisfy the usual axioms, and that they precisely reflect the corresponding operations and relations on the real numbers that are represented. For instance, we have the following.

Lemma 2.19. $+_{\mathbb{R}}$ *respects* $=_{\mathbb{R}}$. *That is,*

$$\mathsf{HA}^\omega \vdash f =_{\mathbb{R}} g \leftrightarrow f +_{\mathbb{R}} h =_{\mathbb{R}} g +_{\mathbb{R}} h \ .$$

And if f, g, h represent the real numbers x, y, z, then $f +_{\mathbb{R}} g =_{\mathbb{R}} h$ if and only if $x + y = z$.

Proof. For the first part, from left to right, we need to show that

$$\forall n (|(\widehat{f +_{\mathbb{R}} h})(n+1) - (\widehat{g +_{\mathbb{R}} h})(n+1)| < 2^{-n}) \ .$$

Since the definition of $+_{\mathbb{R}}$ ensures that $f + h$ is a Cauchy sequence with rate 2^{-n}, $\widehat{f +_{\mathbb{R}} h} =_1 f +_{\mathbb{R}} h$. And so, the above easily follows from $f =_{\mathbb{R}} g$. For the other direction we assume, by the same argument as above, that

$$\forall n (|\widehat{f}(n+2) - \widehat{g}(n+2)| < 2^{-n}) \ .$$

And so, by Lemma 2.18, $f =_{\mathbb{R}} g$.

The second part follows again from basic facts of analysis: If $f +_{\mathbb{R}} g =_{\mathbb{R}} h$, then the respective limits x, y, z of the corresponding (actual) rational sequences must satisfy $x + y = z$. On the other hand, if $x + y = z$, then the (actual) rational sequences given by $f + g$ and h must have the same limit, and so, since $f +_{\mathbb{R}} g$ is defined such that it is Cauchy with rate 2^{-n}, $f +_{\mathbb{R}} g =_{\mathbb{R}} h$ as we saw in Lemma 2.18. ∎

We shall often use basic facts like

- $\leq_{\mathbb{R}}$ is provably reflexive, antisymmetric and transitive.
- $x <_{\mathbb{R}} y \wedge y \leq_R z \to x <_{\mathbb{R}} z$ and $x \leq_{\mathbb{R}} y \wedge y <_{\mathbb{R}} z \to x <_{\mathbb{R}} z$.
- $x <_{\mathbb{R}} y \to x \leq_{\mathbb{R}} y$.

The following expresses a consequence of the "apartness"-interpretation of $<_{\mathbb{R}}$; two numbers being apart if there is a distance between them.

Lemma 2.20.

$$x >_{\mathbb{R}} y \to \exists k(x >_{\mathbb{R}} y +_{\mathbb{R}} 2^{-k}) \ .$$

Proof.

$$\begin{aligned}
x >_{\mathbb{R}} y &\to \exists m(x(m+1) -_{\mathbb{Q}} y(m+1) \geq_{\mathbb{Q}} 2^{-m}) \\
&\to \exists m'(x(m'+1) -_{\mathbb{Q}} y(m'+1) >_{\mathbb{Q}} 2^{-m'}) \qquad (m' = m+1) \\
&\to \exists m', k(x(m'+1) -_{\mathbb{Q}} y(m'+1) >_{\mathbb{Q}} 2^{-m'} +_{\mathbb{Q}} 2^{-k}) \\
&\to \exists k(x >_{\mathbb{R}} y +_{\mathbb{R}} 2^{-k}) \ .
\end{aligned}$$

□

2.4.2 Uniformly Continuous Functions $C[0,1]$

We shall use the notation $x \in_{\mathbb{R}} [0,1]$ to express that x^1 represents a real number in the interval $[0,1]$.

In classical analysis a continuous function $f : [0,1] \to \mathbb{R}$ is known to be uniformly continuous and therefore have a modulus of uniform continuity ω_f^1; ie. a number theoretic function satisfying

$$\forall k^0 \forall x, y \in_{\mathbb{R}} [0,1](|x -_{\mathbb{R}} y|_{\mathbb{R}} <_{\mathbb{R}} 2^{-\omega_f(k)} \to |f(x) -_{\mathbb{R}} f(y)|_{\mathbb{R}} <_{\mathbb{R}} 2^{-k}) \ .$$

When given a uniformly continuous function, we will consider the modulus as part of that data. When we have ω_f, we can get back f from its restriction $f_{\mathbb{Q}} : [0,1] \cap \mathbb{Q} \to \mathbb{R}$. $f_{\mathbb{Q}}$ is of type $0 \to (0 \to 0)$, and therefore it can be coded as a type 1 function. In total, we can represent f as the tuple of $f_{\mathbb{Q}}$ and ω_f — thus, elements in $C[0,1]$, the uniformly continuous functions defined on $[0,1]$, will be represented as type 1 objects.

This choice of representation will serve two purposes. Firstly, it is possible to have every type 1 function represent an element in $C[0,1]$ and thereby quantification over $C[0,1]$ can be reduced to quantification over type 1 objects. Secondly, having access to ω_f allows one to define the supremum, $\sup_{[0,1]}(f)$, of $f \in C[0,1]$ in HA^ω (using at most recursion of type R_0). Both issues are thoroughly described in the appendix of [29].

Chapter 3

Proof Theory

This chapter introduces the proof theoretic tools that will be used later.

Proof theory is a mathematical study of formal mathematical proofs and provability. Two of the main tasks of proof theory is

1. to analyse the proof-theoretic strength of given formal proof systems

2. to investigate what information can be extracted from proofs, other than the truth of the theorem in question.

The proof-theoretic strength of a system is, in its broadest sense, given in terms of the theorems provable in the system. Various classes of theorems have been studied for the purpose of characterising the strength of a proof system. Eg. theorems stating the totality of certain functions, or well-known theorems of analysis.

In the fifties, G. Kreisel formulated the second task mentioned above as follows.

> What more do we know if we have proved a theorem by restricted means than if we merely know that it is true?

Such information — the "more" we know — might be of (semi-) constructive nature though the proof seems non-constructive. The branch of proof theory concerned with the constructive content of actual mathematical proofs is, as mentioned in the introduction, called proof mining.

3.1 Basic Translations

Here we give two simple proof theoretic methods that readily provide useful information on HA. The first, negative translation, gives general insights into HA compared to its classical counterpart PA. We use the other proof translation to prove one concrete result which will shed light on the status of Markov's principle as a semi-classical principle.

3.1.1 Negative Translation

There are several negative translations, which all provide a translation of classical arithmetic into intuitionistic arithmetic. The one we present here is from [39] and has a particularly simple formulation.

Definition 3.1. For a formula A of HA, we define the *negative translation* by $A' :\equiv \neg\neg A^*$, where A^* is inductively defined as

(i) $A^* :\equiv A$, if A is atomic.

(ii) $(A \Box B)^* :\equiv A^* \Box B^*$, if $\Box$ is one of $\vee, \wedge, \rightarrow$.

(iii) $(\exists x A(x))^* :\equiv \exists x A^*(x)$.

(iv) $(\forall x A(x))^* :\equiv \forall x \neg\neg A^*(x)$.

Note that A and A' are classically equivalent.

The main result for the negative translation is the following theorem.

Theorem 3.2 ([11, 12, 39]). *Let A be a formula of* HA*. Then* $\mathsf{PA} \vdash A \iff \mathsf{HA} \vdash A'$.

So any theorem of PA may be stated in a classically equivalent way which is a theorem of HA. In a certain sense, one might therefore say that intuitionistic logic is not weaker than classical logic, but *finer*.

3.1.2 A-Translation

The A-Translation provides an elegant method for proving fundamental closure properties of HA, and has applications in connection with other proof interpretations.

Definition 3.3. For a fixed formula A of HA, we inductively define the formula B^A for any B by

(i) $B^A :\equiv B \vee A$, if B is atomic.

(ii) $(B \Box C)^A :\equiv B^A \Box C^A$, if $\Box$ is one of $\vee, \wedge, \rightarrow$.

(iii) $(QxB(x))^A :\equiv QyB^A(y)$, if Q is one of $\forall$, $\exists$ and y a variable not occurring free in A.

The central property of the A-Translation is the following theorem.

Theorem 3.4 ([9, 10]). *For any formula A of* HA*,*

$$\mathsf{HA} \vdash B \Rightarrow \mathsf{HA} \vdash B^A \ .$$

As a simple, yet important, use of the A-translation, we have give the following theorem.

Theorem 3.5. HA *is closed under the Markov rule, ie.: Let $t[\underline{a}]$ be some term in* HA *with number parameters $\underline{a}$ then*

$$\mathsf{HA} \vdash \neg\neg\exists n(t(n, \underline{a}) = 0) \Rightarrow \mathsf{HA} \vdash \exists n(t(n, \underline{a}) = 0) \ .$$

Proof. We leave out the parameter $\underline{a}$ for the sake of notational simplicity. Assume $\mathsf{HA} \vdash \neg\neg\exists n(t(n) = 0)$. Then by Theorem 3.4,

$$\mathsf{HA} \vdash (\neg\neg\exists n(t(n) = 0))^{\exists m(t(m)=0)} \ .$$

Now, by the definition of $\neg$ and since $\bot \vee A \leftrightarrow A$, we get

$$\mathsf{HA} \vdash \big[\exists n\big(t(n) = 0 \vee \exists m(t(m) = 0)\big) \to \exists n(t(n) = 0)\big] \to \exists n(t(n) = 0 \ ,$$

and so

$$\mathsf{HA} \vdash \big[\big(\exists n(t(n) = 0) \vee \exists m(t(m) = 0)\big) \to \exists n(t(n) = 0)\big] \to \exists n(t(n) = 0 \ .$$

By *modus ponens* and the schema $A \vee A \to A$, we get the result. $\square$

3.2 Realizability

The term "realizability" is used for a family of proof interpretations of which the first, numerical realizability, was introduced by Kleene in [23], intended to make the constructive interpretation of the logical operators explicit.

Kleene realizability interprets "constructive" (in arithmetic) as given by a recursive method, which by Gödel numbering is just a natural number. It therefore considers partial recursive functions. If we instead insist on only realizing formulas by total functions, it is necessary to use higher type recursion. Modified realizability is a realizability interpretation which explores this direction; we present the interpretation below.

3.2.1 Modified Realizability

Modified realizability was introduced in [38] and [37] made references to it, motivating it by a proof of the underivability of Markov's principle.

Definition 3.6. For formulas of HA^ω we define $\underline{x}\ \mathtt{mr}\ A$ by induction on the logical structure of A. $\underline{x}$ is a (possibly empty) tuple of variables not occurring free in A, and the free variables of $\underline{x}\ \mathtt{mr}\ A$ are among those of A and those of $\underline{x}$.

(i) If A is atomic, $\underline{x}\ \mathtt{mr}\ A :\equiv A$, with $\underline{x}$ being the empty tuple.

(ii) $\underline{x}, \underline{y}\ \mathtt{mr}\ (A \wedge B) :\equiv \underline{x}\ \mathtt{mr}\ A \wedge \underline{y}\ \mathtt{mr}\ B$.

(iii) $z^0, \underline{x}, \underline{y}\ \mathtt{mr}\ (A \vee B) :\equiv [(z = 0 \to \underline{x}\ \mathtt{mr}\ A) \wedge (z \neq 0 \to \underline{y}\ \mathtt{mr}\ B)]$.

(iv) $\underline{x}\ \mathtt{mr}\ (A \to B) :\equiv \forall \underline{y}(\underline{y}\ \mathtt{mr}\ A \to \underline{xy}\ \mathtt{mr}\ B)$.

(v) $\underline{x}\ \mathtt{mr}\ (\forall y^\sigma A(y)) :\equiv \forall y(\underline{x}y\ \mathtt{mr}\ A(y))$.

(vi) $z^\sigma, \underline{x}\ \mathtt{mr}\ (\exists y^\sigma A(y)) :\equiv \underline{x}\ \mathtt{mr}\ A(z)$.

In the rest of this section Δ will denote an arbitrary set of $\exists$-free formulas (cf. Definition 2.6).

Theorem 3.7 (Soundness, [54, 3.4.5]).

$$\mathsf{HA}^\omega + \mathsf{AC} + \mathsf{IP}^\omega_{\mathrm{ef}} + \Delta \vdash A \Rightarrow \mathsf{HA}^\omega + \Delta \vdash \underline{t}\ \mathtt{mr}\ A \ ,$$

for some tuple of terms $\underline{t}$ with its free variables among the free variables of A.

Theorem 3.8 (Program Extraction, [54, 3.4.8]). *Let $\forall x^\rho \exists y^\sigma A(x, y)$ be a closed formula of HA^ω. Then*

$$\mathsf{HA}^\omega + \mathsf{AC} + \mathsf{IP}^\omega_{\mathrm{ef}} + \Delta \vdash \forall x^\rho \exists y^\sigma A(x, y) \Rightarrow \mathsf{HA}^\omega + \mathsf{AC} + \mathsf{IP}^\omega_{\mathrm{ef}} + \Delta \vdash \forall x^\rho A(x, t(x)) \ ,$$

where t is a closed term of HA^ω that can be extracted from the proof of $\forall x \exists y A(x, y)$.

3.2.2 Monotone Modified Realizability

The soundness theorem of modified realizability (Theorem 3.7) is concerned with the existence of realizers $\underline{t}$, such that if $\forall \underline{a} A(\underline{a})$ is closed, we have $\forall \underline{a}(\underline{t}(\underline{a})\ \mathtt{mr}\ A(\underline{a}))$. Instead we could look for bounds (in terms of majorizability); ie., terms $\underline{t}'$ such that

$$\exists \underline{x}(\underline{t}' \operatorname{maj} \underline{x} \wedge \forall \underline{a}(\underline{x}(\underline{a})\ \mathtt{mr}\ A(\underline{a})))\ . \tag{3.1}$$

We saw that the class of $\exists$-free formulas had already a modified realizability interpretation. If we just consider extraction of bounds, saying that $A(\underline{a})$ has a monotone modified realizability interpretation if (3.1) is satisfied, then a much larger class of formulas gets satisfied.[1] We have the following theorem.

Theorem 3.9 ([30] and [34]). *Let Γ be a set of closed formulas of the form $\exists \underline{v} \leq_\tau \underline{r} \neg B(\underline{v})$, where $\underline{r}$ is a tuple of closed terms with types given by τ. For a formula $A(x^1, y^\sigma)$ (with only x, y free) with $\deg(\sigma) \leq 2$ we have the following rule*

$$\begin{aligned} &\mathsf{HA}^\omega + \mathsf{AC} + \mathsf{IP}^\omega_{\mathrm{ef}} + \Gamma \vdash \forall x^1 \exists y^\sigma A(x, y) \Rightarrow \\ &\mathsf{HA}^\omega + \mathsf{AC} + \mathsf{IP}^\omega_{\mathrm{ef}} + \Gamma \vdash \forall x^1 \exists y \leq_\sigma t(x) A(x, y)\ , \end{aligned}$$

where t is a closed term of HA^ω.

We could instead let Γ be a set of formulas given with terms provably satisfying their monotone modified realizability interpretation.

3.3 Functional Interpretation

This section introduces Gödel's functional interpretation and a variant known as monotone functional interpretation. Gödel's results were originally published in the journal *Dialectica* as [13], and the interpretation is therefore also known as the Dialectica interpretation.

The relationship between monotone and regular functional interpretation is much like that between monotone and regular modified realizability; instead of actual "realizers", bounds are extracted.

[1]That the class of formulas with a monotone modified realizability interpretation contains that of formulas with a (regular) modified realizability interpretation is not quite obvious, but relies heavily on Theorem 2.5.

3.3.1 Gödel's Functional (Dialectica) Interpretation

Definition 3.10. To each formula A of HA^ω we define a translation $A^D \equiv \exists \underline{x} \forall \underline{y} A_D(\underline{x}, \underline{y})$, where A_D is a quantifier-free formula of HA^ω. The free variables of A_D are among those of A and the (possibly empty) tuples $\underline{x}, \underline{y}$. When we refer to a free variable of A, say z in $A(z)$, then we write $A_D(\underline{x}, \underline{y}, z)$ for A_D.

$(\,)_D$ and $(\,)^D$ are defined simultaneously by induction on the logical structure.

(i) $A^D \equiv A_D \equiv A$ if A is atomic.

Let $A^D \equiv \exists \underline{x} \forall \underline{y} A_D(\underline{x}, \underline{y})$ and $B^D \equiv \exists \underline{u} \forall \underline{v} B_D(\underline{u}, \underline{v})$.

(ii) $(A \wedge B)^D \equiv \exists \underline{x}, \underline{u} \forall \underline{y}, \underline{v} (A_D(\underline{x}, \underline{y}) \wedge B_D(\underline{u}, \underline{v}))$.

(iii) $(A \vee B)^D \equiv \exists z, \underline{x}, \underline{u} \forall \underline{y}, \underline{v} [(z = 0 \rightarrow A_D(\underline{x}, \underline{y})) \wedge (z \neq 0 \rightarrow B_D(\underline{u}, \underline{v}))]$.

(iv) $(A \rightarrow B)^D \equiv \exists \underline{U}, \underline{Y} \forall \underline{x}, \underline{v} (A_D(\underline{x}, \underline{Y}(\underline{x}, \underline{v})) \rightarrow B_D(\underline{U}(\underline{x}), \underline{v}))$.

(v) $(\exists z A(z))^D \equiv \exists z, \underline{x} \forall \underline{y} A_D(\underline{x}, \underline{y}, z)$.

(vi) $(\forall z A(z))^D \equiv \exists \underline{X} \forall z, \underline{y} A_D(\underline{X}(z), \underline{y}, z)$.

The general idea is to use prenexing and AC in all inductive cases to make the formula fit a $\exists\forall$ pattern. This touches upon two issues that need further explanation. The prenexing equivalences are, in general, not all valid in intuitionistic logic, which questions the relationship between a formula and its D-translation. Furthermore, it is not obvious how we choose a particular prenexiation. The first issue will be addressed below in the discussion preceding Theorem 3.13, arguing that actually $A \leftrightarrow A^D$ is provable in a suitable constructive system. To find this system, we will now turn to the second issue by examining the only case of the definition which gives rise to use of prenex rules that are not intuitionistically valid, namely (iv) implication. We consider $A^D \rightarrow B^D$.

$$\begin{array}{cr}
 & (\exists \underline{x} \forall \underline{y} A_D(\underline{x}, \underline{y}) \rightarrow \exists \underline{u} \forall \underline{v} B_D(\underline{u}, \underline{v})) \\
\leftrightarrow & \forall \underline{x} (\forall \underline{y} A_D(\underline{x}, \underline{y}) \rightarrow \exists \underline{u} \forall \underline{v} B_D(\underline{u}, \underline{v})) \\
\overset{\mathsf{IP}^\omega_\forall}{\leftrightarrow} & \forall \underline{x} \exists \underline{u} (\forall \underline{y} A_D(\underline{x}, \underline{y}) \rightarrow \forall \underline{v} B_D(\underline{u}, \underline{v})) \\
\leftrightarrow & \forall \underline{x} \exists \underline{u} \forall \underline{v} (\forall \underline{y} A_D(\underline{x}, \underline{y}) \rightarrow B_D(\underline{u}, \underline{v})) \\
\overset{\mathsf{M}^\omega}{\leftrightarrow} & \forall \underline{x} \exists \underline{u} \forall \underline{v} \exists \underline{y} (A_D(\underline{x}, \underline{y}) \rightarrow B_D(\underline{u}, \underline{v})) \\
\overset{\mathsf{AC}}{\leftrightarrow} & \exists \underline{U}, \underline{Y} \forall \underline{x}, \underline{v} (A_D(\underline{x}, \underline{Y}(\underline{x}, \underline{v})) \rightarrow B_D(\underline{U}(\underline{x}), \underline{v}))
\end{array}$$

The justification of this prenexiation uses the principles M^ω and $\mathsf{IP}^\omega_\forall$, which are not intuitionistically accepted. However, both principles are very weak — we have seen that $\mathsf{IP}^\omega_{\mathrm{ef}}$, which implies $\mathsf{IP}^\omega_\forall$, has a modified realizability interpretation, and Sect. 4.1.1 presents arguments that the type 0 restriction of M^ω, at least in itself, is, in a sense, constructive. That these principles are weak enough to have a constructive interpretation, even when taken together,[2] follows from Theorem 3.12.

Example 3.11. We give three examples. The two first are useful for the use of functional interpretation in connection with negative translation. The third example is one which we will refer to later.

(i) $(\neg A)^D \equiv \exists \underline{Y} \forall \underline{x} \neg A_D(\underline{x}, \underline{Y}(\underline{x}))$.

(ii) $(\neg\neg A)^D \equiv \exists \underline{X} \forall \underline{Y} \neg\neg A_D(\underline{X}(\underline{Y}), \underline{Y}(\underline{X}(\underline{Y})))$.

(iii) Consider $A :\equiv \forall x \exists y \forall z A_0(x, y, z)$, for A_0 quantifier-free. The negative translation A' of A is

$$\neg\neg\forall x \neg\neg\exists y \forall z \neg\neg A_0(x, y, z) \enspace ,$$

which over HA is equivalent to

$$\forall x \neg\neg\exists y \forall z A_0(x, y, z) \enspace ,$$

by the decidability of quantifier-free formulas in HA. We step by step find the Dialectica interpretation $(A')^D$ of A'.

$$[\forall z A_0(x, y, z)]^D \leftrightarrow \forall z A_0(x, y, z) \enspace .$$

$$[\exists y \forall z A_0(x, y, z)]^D \leftrightarrow \exists y \forall z A_0(x, y, z) \enspace .$$

$$[\neg\neg\exists y \forall z A_0(x, y, z)]^D \leftrightarrow \exists Y \forall Z A_0(x, Y(Z), Z(Y(Z))) \enspace .$$

$$[\forall x \neg\neg\exists y \forall z A_0(x, y, z)]^D \leftrightarrow \exists Y \forall x, Z A_0\big(x, Y(x, Z), Z(Y(x, Z))\big) \enspace .$$

Theorem 3.12 (Soundness, [54, 3.5.4–3.5.5]). *For a formula $A(\underline{a})$ of HA^ω with only $\underline{a}$ free, and Π a set of purely universal, closed formulas $\forall x^\rho B_0(x)$ (B_0 quantifier-free of HA^ω) we have the following rule.*

$$\begin{aligned}\mathsf{HA}^\omega + \mathsf{AC} + \mathsf{IP}^\omega_\forall + \mathsf{M}^\omega + \Pi \vdash A(\underline{a}) \Rightarrow& \\ \mathsf{HA}^\omega + \Pi \vdash A_D(\underline{t}(\underline{a}), \underline{y}, \underline{a})& \enspace ,\end{aligned}$$

where $\underline{t}$ is a tuple of closed terms of HA^ω.

[2] Which $\mathsf{IP}^\omega_{\mathrm{ef}}$ and M^ω do not satisfy; [22, 5.4–5.5] discusses this.

The original paper [13] by Gödel had HA in the premise of the soundness theorem. The higher type extension with the principles $\mathsf{IP}^\omega_\forall$ and M^ω appears first in [57]. For some historical remarks on the functional interpretation cf. [53].

For our applications of the functional interpretation (ie. its soundness theorem) it is essential that it is possible to get back the original formula from its D-translation. In this respect, implication is of interest for two reasons. One might hope to give an induction proof of a lemma of the form $\mathsf{H}^\omega \vdash A^D \to A$, for a suitable system H^ω — in our case HA^ω seems reasonable. In all cases except for $A \equiv B \to C$, the induction goes through. But because of the non-monotonicity of implication, one needs the direction $B \to B^D$ as induction hypothesis to prove $(B \to C)^D \to (B \to C)$. But to prove a characterisation lemma of the form $\mathsf{H}^\omega \vdash A^D \leftrightarrow A$ the discussion of prenexiation indicates that we need $\mathsf{M}^\omega + \mathsf{IP}^\omega_\forall$. This can easily be verified.[3]

Theorem 3.13 (Program extraction, [54, 3.7.5]). *Let Π be as in Theorem 3.12 and $A(x^\rho, y^\sigma)$ a formula of HA^ω with only x, y free. Then the following rule holds.*

$$\begin{aligned}&\mathsf{HA}^\omega + \mathsf{AC} + \mathsf{IP}^\omega_\forall + \mathsf{M}^\omega + \Pi \vdash \forall x^\rho \exists y^\sigma A(x,y) \Rightarrow \\ &\mathsf{HA}^\omega + \mathsf{AC} + \mathsf{IP}^\omega_\forall + \mathsf{M}^\omega + \Pi \vdash \forall x A(x, t(x)) \ ,\end{aligned}$$

where t is a closed term that can be extracted from the proof of $\forall x \exists y A(x,y)$.

Proof sketch. We have

$$(\exists y A(x,y))^D \equiv \exists y, \underline{u} \forall \underline{v} A_D(\underline{u}, \underline{v}, x, y) \ .$$

By assumption and the soundness theorem, we get

$$\mathsf{HA}^\omega + \Pi \vdash \forall x, \underline{v} A_D(\underline{t_2}(x), \underline{v}, x, t_1(x)) \ ,$$

hence

$$\mathsf{HA}^\omega + \Pi \vdash \forall x \exists \underline{u} \forall \underline{v} A_D(\underline{u}, \underline{v}, x, t_1(x)) \quad \big(\equiv \forall x [A(x, t_1(x))]^D\big) \ .$$

Since, as discussed above, $\mathsf{HA}^\omega + \mathsf{AC} + \mathsf{IP}^\omega_\forall + \mathsf{M}^\omega \vdash A \leftrightarrow A^D$ we find

$$\mathsf{HA}^\omega + \mathsf{AC} + \mathsf{IP}^\omega_\forall + \mathsf{M}^\omega + \Pi \vdash \forall x A(x, t_1(x)) \ .$$

$\square$

[3]Instead of strengthening the proof system to make it prove $A^D \to A$ for all A, one might consider weakening the class of formulas considered. In [54, 3.6] the formula class Γ_2 addresses this problem.

3.3.2 Monotone Functional Interpretation

As with (monotone) modified realizability, we define monotone functional interpretation by changing the formulation of soundness from extraction of terms $\underline{t}$ which realize the existential quantifier in $(\forall \underline{a} A(\underline{a}))^D$ to extraction of terms $\underline{t}'$ that majorize such a realizer,

$$\exists \underline{X}(\underline{t}' \operatorname{maj} \underline{X} \wedge \forall \underline{a}, \underline{y} A_D(\underline{X}(\underline{a}), \underline{y}, \underline{a})) \ .$$

This allows us to use a large class of ineffective principles in proofs and still be able to extract bounds.

Let Δ be a set of closed formulas of the form $\forall \underline{u}^{\underline{\delta}} \exists \underline{v} \leq_{\underline{\gamma}} \underline{r}(\underline{u}) \forall \underline{w}^{\underline{\tau}} B_0(\underline{u}, \underline{v}, \underline{w})$, where B_0 is quantifier-free and $\underline{r}$ is a tuple of closed terms.

Theorem 3.14 ([30, 3.18], [34]). *Let $A(x^1, y^\sigma)$ be a formula with only x, y free and $\deg(\sigma) \leq 2$. Then the following rule holds.*

$$\begin{aligned} &\mathsf{HA}^\omega + \mathsf{AC} + \mathsf{IP}^\omega_\forall + \mathsf{M}^\omega + \Delta \vdash \forall x^1 \exists y^\sigma A(x, y) \Rightarrow \\ &\mathsf{HA}^\omega + \mathsf{AC} + \mathsf{IP}^\omega_\forall + \mathsf{M}^\omega + \Delta \vdash \forall x^1 \exists y \leq_\sigma t(x) A(x, y) \ , \end{aligned}$$

where t is a closed term of HA^ω that can be extracted from the assumption.

Alternatively, we could have Δ be a set of arbitrary formulas given with terms provably satisfying their monotone functional interpretation.

Chapter 4

Properties of Non-constructive Principles

4.1 Presentations

This section will present three weak classical principles and discuss their strength in increasing order.

4.1.1 Markov's Principle

The Σ^0_1-DNE principle has a special status in constructive reasoning and proof theory, and it is generally known as Markov's principle.

It is well-known that adding either of LEM and DNE to HA allows one to derive exactly all sentences provable in PA. To verify this, it is enough to argue that $\mathsf{PA} = \mathsf{HA} + \mathsf{LEM}$ proves DNE and that $\mathsf{HA} + \mathsf{DNE}$ proves LEM:

$\neg\neg(A \vee \neg A)$ is provable in HA. Hence $\mathsf{HA} \vdash [\neg\neg B \to B] \to A \vee \neg A$, where B is $A \vee \neg A$. Conversely $\mathsf{HA} \vdash A \vee \neg A \to [\neg\neg A \to A]$.

Therefore it seems plausible that Markov's principle is not derivable in HA, which was formally proved in 1959/1962 by Kreisel, see Proposition 4.11. It is therefore reasonable to say that Σ^0_1-DNE is "non-constructive". And indeed, Brouwer and the intuitionists did not accept the principle. Brouwer implicitly advocates his point of view in [6], where he, based on a predicate A for which neither $\neg A$ nor $\neg\neg A$ is known, constructs a real number x such that $\neg(x = 0)$ can be proved but $x < 0 \vee x > 0$ cannot — that is, we can prove $\neg\neg\exists n B(n)$ but not $\exists n B(n)$, refuting Markov's principle.

However, Markov's principle seems rather harmless: If we first establish that it is contradictory that $\forall n \neg A_0(n)$, then we *can*, due to the decidability of quantifier-free formulas in HA, algorithmically find an n such that $A_0(n)$

— we simply try the natural numbers one after the other, it has to terminate. A.A. Markov (1903–1979) argued this way in [44] and represents a branch of constructive mathematics accepting the principle.

Even though the discussion above shows Markov's principle to be somewhere in the wastelands between constructive and classical reasoning, we can make precise statements about its proof-theoretic strength and weakness if we switch from the subjective area of constructivism to a (much more) objective one, namely HA.

The closure of HA under Markov's rule given in Theorem 3.5 describes the weakness of Markov's principle. Notice that it seems to refute Brouwer's refutation: If we can prove $\neg(x = 0)$ then we can also prove $x < 0 \lor x > 0$. But actually, it only shows that Brouwer did not argue in HA.

Markov's argument and the closure property strongly suggest that from a pragmatic point of view Markov's principle is harmless. And certainly, it allows program extraction:

Theorem 4.1. *Let $A(a, b)$ be an arbitrary formula of HA containing only a and b free. Then*

$$\mathsf{HA} + \Sigma^0_1\text{-}\mathsf{DNE} \vdash \forall a \exists b A(a, b) \Rightarrow \mathsf{HA}^\omega + \mathsf{AC} + \mathsf{M}^\omega + \mathsf{IP}^\omega_\forall \vdash \forall a A(a, t(a)) \;\;,$$

where t is a closed term of HA^ω that can be extracted from the proof of $\forall a \exists b A(a, b)$.

Remark 4.2. The result follows from the more general Theorem 3.13 since $\mathsf{HA} + \Sigma^0_1\text{-}\mathsf{DNE}$ can be viewed as a subsystem of $\mathsf{HA}^\omega + \mathsf{M}^\omega$.

The strength of the system used to prove that the extracted term actually realizes b in $\forall a \exists b A(a, b)$ is not essential to us — we only need to know that it is true.

So what we mean by *program extraction* is that from a proof of a $\forall x \exists y$ statement it is possible to extract a recursive function realizing y given x.

4.1.2 Π^0_1-LEM

Further towards non-constructivity but still on the borderline is the logical principle Π^0_1-LEM. Neither the intuitionists nor the Russian constructivist school accepted the principle, which is very reasonable since Π^0_1-LEM directly allows us to decide whether the language accepted by some Turing machine is empty, and this is known not to be a recursive task. The point is proved in the following proposition.

Proposition 4.3. $\mathsf{HA} + \Pi^0_1\text{-}\mathsf{LEM}$ *does not allow program extraction. In fact, recursive realizers do not exist.*

Proof. Assume that it were possible to extract computable realizers for $\forall a \exists b A(a,b)$ statements proved in $\mathsf{HA} + \Pi^0_1\text{-}\mathsf{LEM}$. Let

$$A(a,b) :\equiv (b = 0 \to \forall n \neg T(a,(n)_0,(n)_1) \wedge b \neq 0 \to \neg \forall n \neg T(a,(n)_0,(n)_1)) \ .$$

Then $\forall a \exists b A(a,b)$ can be proved using $\Pi^0_1\text{-}\mathsf{LEM}$. We would then get a recursive function f such that

$$\forall a (f(a) = 0 \leftrightarrow \forall n \neg T(a,(n)_0,(n)_1)) \ ,$$

which contradicts the undecidability of the "empty" problem. □

However, using $\Pi^0_1\text{-}\mathsf{LEM}$ we can still extract bounds using monotone modified realizability:

Theorem 4.4 ([30]). *Let $A(a,b)$ be an arbitrary formula containing only a and b free. Then*

$$\mathsf{HA} + \Pi^0_1\text{-}\mathsf{LEM} \vdash \forall a \exists b A(a,b) \Rightarrow \\ \mathsf{PA}^\omega + \mathsf{AC} \vdash \forall a \exists b \leq t(a) A(a,b) \ ,$$

where t is a closed term of HA^ω that can be extracted from the proof.

Remark 4.5. The result follows from Theorem 3.9. Note however that the set Γ from Theorem 3.9 must consist of *closed* formulas, whereas instances of $\Pi^0_1\text{-}\mathsf{LEM}$ are allowed to have number parameters. The proof from [30] even shows that $\mathsf{PA}^\omega + \mathsf{AC}$ can be replaced by a much weaker system.

So by *bound extraction* we mean the property that one from a proof of a $\forall a \exists b$ statement can extract a function f that recursively in a gives a bound on b; $\forall a \exists b \leq f(a)$.

The fact that it is useful to get merely a bound and not an actual realizer, is an observation made in [27], the reason being that properties considered in applications of proof mining often are monotone (or can be made so) which means that a bound is actually a realizer.[1]

4.1.3 Σ^0_1-LEM

We now turn to a logical principle that has none of the two properties mentioned above — it neither preserves program extraction nor bound extraction — namely the $\Sigma^0_1\text{-}\mathsf{LEM}$ principle. The intuitive reason for the lack of bound extraction is that whereas the existential information of $\Pi^0_1\text{-}\mathsf{LEM}$ corresponds to a binary oracle choice between the two disjuncts, the information

[1] For more details on such applications refer to [34].

of Σ^0_1-LEM is a full countable oracle choice that picks a counterexample to $\forall$ or tells us that no such counterexample exists. We have the following proposition.

Proposition 4.6. Σ^0_1-LEM *does not allow bound extraction.*

Proof. Since Σ^0_1-IP follows from Σ^0_1-LEM, we have

$$\mathsf{HA} + \Sigma^0_1\text{-}\mathsf{LEM} \vdash \forall e \exists m (\exists n T(e,e,n) \to T(e,e,m)) \enspace .$$

If we could extract a computable bound we would get a term t of HA^ω such that,

$$\forall e \exists m \leq t(e) (\exists n T(e,e,n) \to T(e,e,m)) \enspace ,$$

which would solve the halting problem effectively. ☐

Σ^0_1-LEM does, however, have a limit recursive realizer, as mentioned in Chap. 1.

4.1.4 Concluding Connection

A final, pragmatic reason not to include Markov's Principle in the constructive logical core system is based on the following lemma:

Lemma 4.7. *For any term t of* HA

$$\mathsf{HA} + \Sigma^0_1\text{-}\mathsf{DNE} \vdash \Sigma^0_1\text{-}\mathsf{LEM}(t) \leftrightarrow \Pi^0_1\text{-}\mathsf{LEM}(\overline{\mathrm{sg}}(t)) \enspace .$$

Proof. Left to right is even provable without DNE. For the other direction we aim at $\neg\exists n(t(n) = 0) \vee \exists n(t(n) = 0)$. From Π^0_1-LEM we already have $\forall n \neg(t(n) = 0) \vee \neg\forall n \neg(t(n) = 0)$, which over HA is equivalent to $\neg\exists n(t(n) = 0) \vee \neg\neg\exists n(t(n) = 0)$. ☐

That is, $\mathsf{HA} + \Sigma^0_1$-DNE is not as "stable" as HA, in the sense that adding Π^0_1-LEM no longer allows bound extraction, since the resulting system actually is equivalent to $\mathsf{HA} + \Sigma^0_1$-LEM. Or in other words, HA gives rise to more interesting semi-constructive systems in that it distinguishes more principles than $\mathsf{HA}+\Sigma^0_1$-DNE. Also, for this reason there are useful proof interpretations, like (monotone) modified realizability, that actually do not satisfy Σ^0_1-DNE.

To summarise, Markov's principle is in many ways harmless on its own. But there is good reason to not have it in the formal constructive system.

4.2 Generalisations and Relations

We now aim at ordering the various logical principles introduced in Definitions 2.8–2.12 for $n = 1, 2$ — to the extend that the principles are comparable. The result is the hierarchy depicted below. The main theorem of [1] is a generalised version of Theorem 4.8 considering all n. The proof of the general theorem is beyond the scope of this project, but restricting ourselves to the cases $n = 1, 2$, we can give the proof theoretically interesting arguments that are the core of the proof in [1], without worrying about the generalisation. Furthermore, for this project we shall only refer to Theorem 4.8, and not the general hierarchy.

Theorem 4.8 ([1]). *In the following diagram,* $\longrightarrow\!\!\shortmid$ *denotes an implication over* HA *that cannot be reversed and* $\not\twoheadrightarrow$ *refers to unprovability in* HA.

$$\begin{tikzcd}
 & \Sigma^0_2\text{-}\mathsf{LEM} \arrow[dl, "(13)"'] \arrow[d, "(14)"] \\
\Sigma^0_2\text{-}\mathsf{DNE} \arrow[dr, "(10)", "/" marking] \arrow[ddr, "(8)"'] & \Pi^0_2\text{-}\mathsf{LEM} \arrow[l, "(12)"', "/" marking] \arrow[d, "(11)"] \\
 & \Sigma^0_2\text{-}\mathsf{LLPO} \arrow[d, "(9)"] \\
 & \Sigma^0_1\text{-}\mathsf{LEM} \arrow[dl, "(6)"'] \arrow[d, "(7)"] \\
\Sigma^0_1\text{-}\mathsf{DNE} \arrow[dr, "(3)", "/" marking] \arrow[ddr, "(1)"'] & \Pi^0_1\text{-}\mathsf{LEM} \arrow[l, "(5)"', "/" marking] \arrow[d, "(4)"] \\
 & \Sigma^0_1\text{-}\mathsf{LLPO} \arrow[d, "(2)"] \\
 & \Sigma^0_0\text{-}\mathsf{LEM}
\end{tikzcd}$$

Remark 4.9. To make it explicit, $\mathsf{P}_1 \longrightarrow\!\!\shortmid \mathsf{P}_2$ is short for $\mathsf{HA} + \mathsf{P}_1 \vdash \mathsf{P}_2$ *and* $\mathsf{HA} + \mathsf{P}_2 \nvdash \mathsf{P}_1$. And $\mathsf{P}_1 \not\twoheadrightarrow \mathsf{P}_2$ is short for $\mathsf{HA} + \mathsf{P}_1 \nvdash \mathsf{P}_2$.

Proposition 4.10. *We gather the positive parts of* (1)*,* (2)*,* (4)*,* (6)*,* (7) *and* (8)*,* (9)*,* (11)*,* (13)*,* (14)*.*

Proof. (1) and (2) are obvious since Σ^0_0-LEM is provable in HA.

(4): We consider the instance Σ^0_1-$\mathsf{LLPO}(t_1, t_2)$. With Π^0_1-LEM we can prove $\Pi^0_1(\overline{\mathrm{sg}}(t_1)) \vee \neg\Pi^0_1(\overline{\mathrm{sg}}(t_1))$ and the same for t_2. The only problematic case is

$$\neg\Pi^0_1(\overline{\mathrm{sg}}(t_1)) \wedge \neg\Pi^0_1(\overline{\mathrm{sg}}(t_2)) \ ,$$

which in HA is equivalent to

$$\neg\neg\Sigma^0_1(t_1) \wedge \neg\neg\Sigma^0_1(t_2) \ .$$

Pulling out the double negations we get

$$\neg\neg(\Sigma^0_1(t_1) \wedge \Sigma^0_1(t_2)) \ ,$$

contradicting the premise of Σ^0_1-LLPO, and we therefore also in this case, by $\perp \to A$, get $\Pi^0_1(\overline{\mathrm{sg}}(t_1)) \vee \Pi^0_1(\overline{\mathrm{sg}}(t_2))$.

(6) and (7) are straightforward.
(8): The argument is essentially the one sketched in the beginning of Sect. 4.1.1. Let the Σ^0_1-LEM instance be given by the term t. In HA we can prove $\neg\neg(\exists k(t(k) = 0) \vee \forall l \neg(t(l) = 0))$. This implies

$$\neg\neg\exists k \forall l[t(k) = 0 \vee \neg t(l) = 0] \ .$$

So by Σ^0_2-DNE we get

$$\exists k \forall l[t(k) = 0 \vee \neg t(l) = 0] \ ,$$

and then also

$$\exists k(t(k) = 0) \vee \forall l \neg(t(l) = 0) \ .$$

(9): Assuming Σ^0_2-LLPO, we again aim at

$$\exists l(t(l) = 0) \vee \forall k \neg(t(k) = 0) \ .$$

Let $t_1(k, l) := \overline{\mathrm{sg}}(t(l))$ and $t_2(k, l) = t(k)$. Then

$$\begin{aligned} \exists k \forall l(t_1(k, l) = 0) \wedge \exists k \forall l(t_2(k, l) = 0) &\to \forall l \neg(t(l) = 0) \wedge \exists k(t(k) = 0) \\ &\to \perp \ . \end{aligned}$$

By Σ^0_1-$\mathsf{LLPO}(t_1, t_2)$ we thus get

$$\forall k \exists l \neg\neg(t(l) = 0) \vee \forall k \exists l \neg(t(k) = 0) \ ,$$

hence

$$\exists l(t(l) = 0) \lor \forall k \neg (t(k) = 0) \ .$$

(11): Assume Π^0_2-LEM and let t_1 and t_2 be terms satisfying the premise of Σ^0_2-LLPO. As in (4) the only non-trivial case is

$$\neg \forall k \exists l \neg (t_1(k,l) = 0) \land \neg \forall k \exists l \neg (t_2(k,l) = 0) \ .$$

Using Σ^0_1-DNE[2] we can prove

$$\forall k \neg\neg \exists l \neg (t_1(k,l) = 0) \to \forall k \exists l \neg (t_1(k,l) = 0) \ .$$

Hence we get

$$\neg \forall k \neg\neg \exists l \neg (t_1(k,l) = 0) \land \neg \forall k \neg\neg \exists l \neg (t_2(k,l) = 0),$$

which is equivalent to

$$\neg\neg \exists k \forall l (t_1(k,l) = 0) \land \neg\neg \exists k \forall l (t_2(k,l) = 0) \ .$$

This contradicts the assumption on t_1, t_2, so we get $\perp$ and are done.

(13) is straightforward.
(14): From Σ^0_1-LEM we get

$$\exists k \forall l \neg (t_1(k,l) = 0) \lor \neg \exists k \forall l \neg (t_1(k,l) = 0).$$

Intuitionistically valid manipulations then give

$$\begin{aligned} &\exists k \neg \exists l (t_1(k,l) = 0) \lor \neg \exists k \neg \exists l (t_1(k,l) = 0) \\ \to\ &\neg \forall k \exists l (t_1(k,l) = 0) \lor \forall k \neg\neg \exists l (t_1(k,l) = 0) \ . \end{aligned}$$

And so, by Σ^0_1-DNE, we get the desired Π^0_1-LEM instance. $\square$

The first negative result we consider is (1), which states that Σ^0_1-DNE is not derivable in HA and it is proved in the following proposition. Though the negative part of (1) follows from (2), (4) and (5), we give the proof because it was one of the first results of its kind, and its basic idea is shared by the other negative results that we present below.

[2]It is easy to verify directly that Π^0_2-LEM implies Σ^0_1-DNE.

Proposition 4.11 (Kreisel [37, 38]).

$$\mathsf{HA} \nvdash \Sigma^0_1\text{-}\mathsf{DNE} \ .$$

Proof. Assume that

$$\mathsf{HA} \vdash \forall x(\neg\neg\exists y T(x,x,y) \rightarrow \exists y T(x,x,y)) \ ,$$

then using modified realizability we get a closed term t of HA^ω such that

$$\mathsf{HA}^\omega \vdash \forall x(\neg\neg\exists y T(x,x,y) \leftrightarrow T(x,x,t(x))) \ ,$$

which would solve the halting problem, contradicting the fact that t represents a recursive function. $\square$

Next we consider a relativised version of the proposition above, which shows the negative part of Theorem 4.8 (8). There is a completely analogous result on a level higher — if we strengthen our proof system by adding Σ^0_1-LEM, and correspondingly strengthen Σ^0_1-DNE to Σ^0_2-DNE, the proof from above still goes through. Again, the result follows from (9), (11) and (12), but it is included to introduce the relativising technique in a simple case.

Lemma 4.12 ([1]).

$$\mathsf{HA} + \Sigma^0_1\text{-}\mathsf{LEM} \nvdash \Sigma^0_2\text{-}\mathsf{DNE} \ .$$

Proof. Suppose for a term $s[a]$ with only a free that

$$\mathsf{HA} + \Sigma^0_1\text{-}\mathsf{LEM} \vdash \forall a \Sigma^0_2\text{-}\mathsf{DNE}(s[a]) \ .$$

We introduce the ε-axiom (in the sense of [18]),

$$(\varepsilon) \ \ t(x,b) = 0 \rightarrow t(g(b),b) = 0 \ ,$$

to get

$$\mathsf{HA}[g] + (\varepsilon) \vdash \forall b(\exists x(t(x,b)=0) \leftrightarrow t(g(b),b)=0) \ .$$

So $\mathsf{HA}[g] + (\varepsilon) \vdash \forall a \Sigma^0_2\text{-}\mathsf{DNE}(s[a])$ and therefore, by modified realizability,[3] we get a term t representing a functional recursive in g, such that

$$\mathsf{HA}[g]^\omega + (\varepsilon) \vdash \forall a(\exists x \forall y(s(a,x,y)=0) \leftrightarrow \forall y(s(a,t[g](a),y)=0)) \ .$$

[3] The ε-axiom is existential-free and therefore realizes itself.

With a new ε-axiom, (ε'), capturing this last $\forall y$ quantifier and a fresh function symbol h, we get a term t' that recursively in g and h decides the Σ^0_2 statement:

$$\mathsf{HA}[g,h]^\omega + (\varepsilon) + (\varepsilon') \vdash \forall a(\exists x \forall y(s(a,x,y)=0) \leftrightarrow t'[g,h](a)=0) \ .$$

By choosing s to be the matrix of a complete Σ^0_2-relation (cf. [48, 16.1] we get a contradiction to the fact that such a relation is not Turing-reducible to any Σ^0_1-relation.

□

The proof of the next lemma — which is (3) and hence, a fortiori, the remaining part of (2) — explains why Σ^0_1-LLPO (with function parameters) is sometimes (in [56]) called SEP; a constructive interpretation of the principle gives an oracle for separating disjoint, recursively enumerable sets.

Lemma 4.13 ([1]).

$$\mathsf{HA} + \Sigma^0_1\text{-}\mathsf{DNE} \nvdash \Sigma^0_1\text{-}\mathsf{LLPO} \ .$$

Proof. We have

$$\neg(\exists y(T(x,x,y) \wedge U(y)=0) \wedge \exists y(T(x,x,y) \wedge U(y)=1)) \ ,$$

so if we assume Σ^0_1-LLPO to be provable in $\mathsf{HA} + \Sigma^0_1$-DNE, we would get

$$\forall x(\forall y \neg(T(x,x,y) \wedge U(y)=0) \vee \forall y \neg(T(x,x,y) \wedge U(y)=1)) \ .$$

Functional interpretation would then provide a term t of HA^ω such that (arguing in $\mathsf{HA}^\omega + \mathsf{AC} + \mathsf{IP}^\omega_\forall + \mathsf{M}^\omega$)

$$\begin{aligned} t(x) = 0 &\to \neg\exists y(T(x,x,y) \wedge U(y)=0) \quad \text{and} \\ t(x) \neq 0 &\to \neg\exists y(T(x,x,y) \wedge U(y)=1) \ . \end{aligned}$$

Let z be a code of the recursive function represented by $\mathrm{sg}(t)$. Then

$$\begin{aligned} t(z) = 0 &\to \exists y(T(z,z,y) \wedge U(y)=0) && \mathrm{sg}(t) \text{ is total and so } \{z\}(z)\downarrow. \\ &\to t(z) \neq 0 && \text{by the implication above.} \end{aligned}$$

and analogously for $t(z) \neq 0$:

$$\begin{aligned} t(z) \neq 0 &\to \exists y(T(z,z,y) \wedge U(y)=1) \\ &\to t(z) = 0 \ , \end{aligned}$$

which is a contradiction. □

Note that the just proved Theorem 4.8 (3) in combination with the positive part of (4) provides Σ^0_1-$\mathsf{DNE} \not\to \Pi^0_1$-$\mathsf{LEM}$.

The next lemma proves the remaining part of (4). In fact we show a stronger result, namely that both Σ^0_1-DNE and Σ^0_1-LLPO together are still not enough to prove Π^0_1-LEM. The first step is to note that in the presence of Σ^0_1-DNE, Π^0_1-LEM is lifted to Σ^0_1-LEM. As mentioned in Sect. 4.1.3 the latter is of a very different nature — the non-constructivity of Σ^0_1-LLPO is a binary (therefore bounded) one whereas that of Σ^0_1-LEM is in a sense unbounded. It seems unreasonable to expect that the harmless Σ^0_1-DNE and Σ^0_1-LLPO, that does not have any double negated $\exists$ quantifiers, should give rise to such unboundedness. The following shows this intuition to be true.

Lemma 4.14 ([34][4]).

$$\mathsf{HA} + \Sigma^0_1\text{-}\mathsf{LLPO} + \Sigma^0_1\text{-}\mathsf{DNE} \nvdash \Pi^0_1\text{-}\mathsf{LEM} \ .$$

Proof. Assume that $\mathsf{HA} + \Sigma^0_1$-$\mathsf{LLPO} + \Sigma^0_1$-$\mathsf{DNE} \vdash \Pi^0_1$-$\mathsf{LEM}$. Then we can also derive Σ^0_1-LEM and get the following Σ^0_1-IP conclusion for $\forall a(\exists n T(a,a,n) \to \exists n T(a,a,n))$,

$$\forall a \exists n(\exists m T(a,a,m) \to T(a,a,n)) \ .$$

It is easily checked that Σ^0_1-LLPO follows from a sentence with a simple monotone functional interpretation. For instance, define

$$\tilde{t}_1(n) := \begin{cases} 1 & \text{if } t_1(n) = 0 \wedge \forall m \leq n(t_2(n) \neq 0) \\ t_1(n) & \text{otherwise} \ , \end{cases}$$

recursively in t_t, t_2, and analogously for $\tilde{t}_2$. Then $\tilde{t}_1, \tilde{t}_2$ always satisfy the premise of Σ^0_1-LLPO and if t_1, t_2 already did then $\tilde{t}_1 = t_1$ and $\tilde{t}_2 = t_2$. So by equivalently writing the disjunction in the conclusion of Σ^0_1-LLPO in the form $\exists k \leq 1(k = 0 \to A \wedge k = 1 \to B)$ we have a sentence whose monotone functional interpretation is satisfied by the constant 1 function in the parameters. Furthermore, Σ^0_1-DNE has a monotone functional interpretation.

Thus, by monotone functional interpretation we get a term t of HA^ω satisfying

$$\mathsf{HA}^\omega \vdash \forall a \exists n \leq t(a)(\exists m T(a,a,m) \to T(a,a,n)) \ ,$$

which could be used to solve the special halting problem. ∎

We now prove (5) which strengthens Lemma 4.11, and thereby we also show that Σ^0_1-$\mathsf{LLPO} \not\to \Sigma^0_1$-$\mathsf{DNE}$.

[4] In the chapter on monotone functional interpretation.

Lemma 4.15 ([34][5]).

$$\mathsf{HA} + \Pi^0_1\text{-}\mathsf{LEM} \nvdash \Sigma^0_1\text{-}\mathsf{DNE} \ .$$

Proof. If $\mathsf{HA} + \Pi^0_1\text{-}\mathsf{LEM} \vdash \Sigma^0_1\text{-}\mathsf{DNE}$, then also $\mathsf{HA} + \Pi^0_1\text{-}\mathsf{LEM} \vdash \Sigma^0_1\text{-}\mathsf{LEM}$ by Lemma 4.7. Since the monotone modified realizability interpretation of $\Pi^0_1\text{-}\mathsf{LEM}$ is satisfied by the constant 1 function (in the parameters), we get a contradiction as in the proof of Lemma 4.14. □

The next lemma is a relativised variant of lemma 4.14. The relation between these two is much like the one between Propositions 4.11 and 4.12.

Lemma 4.16.

$$\mathsf{HA} + \Sigma^0_2\text{-}\mathsf{LLPO} + \Sigma^0_2\text{-}\mathsf{DNE} \nvdash \Pi^0_2\text{-}\mathsf{LEM} \ .$$

Proof. We first prove that $\mathsf{HA} + \Pi^0_2\text{-}\mathsf{LEM} + \Sigma^0_2\text{-}\mathsf{DNE} \vdash \Sigma^0_2\text{-}\mathsf{LEM}$.

$$\begin{aligned}
&\forall m \exists n \neg A_0(m,n) \vee \neg \forall m \exists n \neg A_0(m,n) \\
\to\ & \forall m \neg \forall n A_0(m,n) \vee \neg \forall m \exists n \neg A_0(m,n) \\
\to\ & \neg \exists m \forall n A_0(m,n) \vee \neg \forall m \exists n \neg A_0(m,n) \\
\overset{*}{\to}\ & \neg \exists m \forall n A_0(m,n) \vee \neg \forall m \neg\neg \exists n \neg A_0(m,n) \\
\to\ & \neg \exists m \forall n A_0(m,n) \vee \neg\neg \exists m \forall n A_0(m,n) \\
\overset{**}{\to}\ & \neg \exists m \forall n A_0(m,n) \vee \exists m \forall n A_0(m,n) \ ,
\end{aligned}$$

where $(*)$ follows from $\Sigma^0_1\text{-}\mathsf{DNE}$ and $(**)$ from $\Sigma^0_2\text{-}\mathsf{DNE}$.

As in the proof of Lemma 4.12 we define an ε-axiom for the new function symbol g to get

$$\mathsf{HA}[g] + (\varepsilon_B) \vdash \forall m (\exists n \neg B_0(m,n) \to \neg B_0(m, g_B(m))) \ ,$$

and similarly for C_0 with (ε_C) and g_C.

We now prove that using g_B, g_C we get $\Sigma^0_2\text{-}\mathsf{LLPO}$ from $\Sigma^0_1\text{-}\mathsf{LLPO}[g_B, g_C]$, where $\mathsf{S}[\underline{f}]$ for some schema S denotes the schema of which instances are allowed to be given by terms of the enriched language of $\mathsf{HA}[\underline{f}]$. Note that when we just write S in the context $\mathsf{HA}[\underline{f}]$ we still only consider instances of HA.

Let the considered $\Sigma^0_2\text{-}\mathsf{LLPO}$ instance be given by quantifier-free B_0, C_0 satisfying

$$\neg(\exists m \forall n B_0(m,n) \wedge \exists m \forall n C_0(m,n)) \ . \tag{4.1}$$

[5]In the chapter on monotone modified realizability.

By $(\varepsilon_B) + (\varepsilon_C)$, (4.1) is equivalent to

$$\neg(\exists m B_0(m, g_B(m)) \wedge \exists m C_0(m, g_C(m))) \ ,$$

which, assuming Σ^0_1-$\mathsf{LLPO}[g_B, g_C]$, gives,

$$\forall m \neg B_0(m, g_B(m)) \vee \forall m \neg C_0(m, g_C(m)) \ .$$

Hence,

$$\forall m \exists n \neg B_0(m, n) \vee \forall m \exists n \neg C_0(m, n) \ .$$

That is, the oracles g_B, g_C lift Σ^0_1-$\mathsf{LLPO}[g_B, g_C]$ to Σ^0_2-LLPO. By similar arguments Σ^0_1-$\mathsf{DNE}[g_B, g_C]$ can be seen lifted to Σ^0_2-DNE. So we have proved that if $\mathsf{HA} + \Sigma^0_2$-$\mathsf{LLPO} \vdash \Pi^0_2$-$\mathsf{LEM}$, then

$$\mathsf{HA}[g_B, g_C] + (\varepsilon_B) + (\varepsilon_C) + \Sigma^0_1\text{-}\mathsf{DNE}[g_B, g_C] + \Sigma^0_1\text{-}\mathsf{LLPO}[g_B, g_C] \vdash \Sigma^0_2\text{-}\mathsf{LEM} \ .$$

Now using monotone functional interpretation[6] we find, for any quantifier free A_0, a term t of $\mathsf{HA}^\omega[g_B, g_C]$ such that

$$\forall k \exists m' \leq t[g_B, g_C](k) \forall n' (\exists m \forall n A_0(m, n) \to A_0(m', n')) \ .$$

Hence

$$\forall k (\exists m \forall n A_0(m, n) \leftrightarrow \exists m' \leq t[g_B, g_C](k) \forall n' A_0(m', n') \ .$$

We then use g_B, g_C to try the $t[g_B, g_C](k)$ different possibilities and get a new term s of $\mathsf{HA}^\omega[g_b, g_C]$ that recursively in the halting problem decides a arbitrary Σ^0_2 statement:

$$\forall k (\exists m \forall n A_0(m, n) \leftrightarrow s[g_B, g_C](k) = 0) \ .$$

By choosing A_0 as the matrix of a complete Σ^0_2-relation, this gives a contradiction. □

To finish the proof of Theorem 4.8 we show the following two lemmata, which relativise Lemma 4.13 respectively Lemma 4.15.

Lemma 4.17.

$$\mathsf{HA} + \Sigma^0_2\text{-}\mathsf{DNE} \nvdash \Sigma^0_2\text{-}\mathsf{LLPO} \ .$$

[6]The ε-axioms are purely universal, and therefore their own functional interpretation. Note also that the interpretations of the schemata Σ^0_1-$\mathsf{LLPO}[g_B, g_C]$ and Σ^0_1-$\mathsf{DNE}[g_B, g_C]$ are easily satisfied.

Proof. For notational simplicity we denote the new function symbols as just one function symbol g and the ε-axioms by (ε). From Post's Theorem (cf. [48, 14.8]), we know that

$$\neg(\exists y(T^g(x,x,y) \wedge U(y) = 0) \wedge \exists y(T^g(x,x,y) \wedge U(y) = 1)) \ ,$$

is equivalent (classically) to a relation of the form $\neg(\Sigma^0_2(t_1) \wedge \Sigma^0_2(t_2))$, for terms t_1, t_2 of HA.[7] Assume that Σ^0_2-LLPO were derivable from Σ^0_2-DNE, hence from Σ^0_1-$\mathsf{DNE}[g]$ (in $\mathsf{HA}[g] + (\varepsilon)$). Applying functional interpretation to the Σ^0_2-LLPO conclusion would then give a contradiction as in the proof of Lemma 4.13. □

Lemma 4.18.

$$\mathsf{HA} + \Pi^0_2\text{-}\mathsf{LEM} \nvdash \Sigma^0_2\text{-}\mathsf{DNE} \ .$$

Proof. If $\mathsf{HA} + \Pi^0_2\text{-}\mathsf{LEM} \vdash \Sigma^0_2\text{-}\mathsf{DNE}$, then also $\mathsf{HA} + \Pi^0_2\text{-}\mathsf{LEM} \vdash \Sigma^0_2\text{-}\mathsf{LEM}$. And so, arguing as in the proof of Lemma 4.16 (with the notational simplification from Lemma 4.17), $\mathsf{HA}[g] + (\varepsilon) + \Pi^0_1\text{-}\mathsf{LEM}[g] \vdash \Sigma^0_2\text{-}\mathsf{LEM}$. Using monotone modified realizability we, again as in Lemma 4.16, get a contradiction. □

4.3 Reflection on the Principles

This section looks at the hierarchy of Theorem 4.8 once again. It will focus on two things — the methods used to differentiate the levels in the hierarchy, and what more these methods do, than merely separating the levels. The discussion will restrict itself to the principles for $n = 1$, but most of it has a relativised (oracle) interpretation on the higher levels.

4.3.1 Interpreting the Proof of Theorem 4.8

In the proof of Theorem 4.8 the basic common idea, when separating two principles, was to find a proof interpretation which satisfied one principle but not the other. Looking through the lemmata one finds that we have used four different interpretations — two pairs of interpretations actually.

The first pair is the modified, and monotone modified, realizability. Modified realizability satisfies none of the principles we consider. We used this to prove that Σ^0_1-DNE was not derivable in HA. Monotone modified realizability satisfies two of our principles; Σ^0_1-LLPO and Π^0_1-LEM, and it therefore, in particular, separates Σ^0_1-DNE from Π^0_1-LEM.

[7] Where we have adopted the oracle notation for the T-predicate, T^g, from [48].

The second pair is functional, and monotone functional, interpretation. Regular functional interpretation satisfies Σ^0_1-DNE, and none of the other principles. The corresponding monotone interpretation satisfies one more of our principles, namely Σ^0_1-LLPO. It does not satisfy Π^0_1-LEM, for it looks enough into the formulas to satisfy Σ^0_1-DNE and hence does not distinguish between Π^0_1-LEM and Σ^0_1-LEM (Lemma 4.7); and we know that Σ^0_1-LEM does not have a monotone functional interpretation (Proposition 4.6).

4.3.2 What More Do we Know

Program Extraction. In Sect. 4.1 we discussed the three logical principles Σ^0_1-DNE, Π^0_1-LEM and Σ^0_1-LEM. We saw that only Σ^0_1-DNE allows program extraction. Sect 4.2 considered also the Σ^0_1-LLPO principle. A first step to fitting in this principle in the discussion from Sect 4.1 is in the proof of Lemma 4.13, which shows that Σ^0_1-LLPO does not preserve the program extraction property.

Bounds. As is the case with Π^0_1-LEM, the positive information in the conclusion of Σ^0_1-LLPO is binary, and it allows bound extraction (cf. Theorem 4.4), since it is weaker than Π^0_1-LEM by Proposition 4.10.

But the nature of Σ^0_1-LLPO is also quite different from that of Π^0_1-LEM. It might happen that in the conclusion of Σ^0_1-LLPO, both disjuncts are true; in such a case, an imagined realizer for the principle would be free to chose either one, since the principle itself reveals no restrictions. Π^0_1-LEM does not have this nondeterministic feature, since the two disjuncts in Π^0_1-LEM cannot both hold.

This difference is used in [1, Theorem 3.14] by the so-called Lifschitz realizability interpretation (cf. [55, Sect. 5]) for which Σ^0_1-LLPO is among the self-realizing formulas — which for modified realizability are the $\exists$-free formulas. The proof given above uses instead the fact that on the one hand Σ^0_1-DNE exposes a positive $\exists$-quantifier in Π^0_1-LEM, but on the other it has no real effect on Σ^0_1-LLPO.

The extra strength of Π^0_1-LEM has a price. As explained above, Π^0_1-LEM allows bound extraction, but also adding Σ^0_1-DNE is not conservative with respect to this property. This, however, is the case for Σ^0_1-LLPO, since Σ^0_1-LLPO+ Σ^0_1-DNE has a monotone functional interpretation. So proof theoretically one can *either* use Π^0_1-LEM *or* use Σ^0_1-LLPO + Σ^0_1-DNE and in both cases preserve the bound extraction property.

Limits Beyond the Bounds. When using principles as strong as Σ^0_1-LEM, it is not possible to extract bounds (in the sense of Proposition 4.6). Yet, as explained in Sect. 2.1.1, Σ^0_1-LEM and even Σ^0_2-DNE have limit recursive realizers, and therefore some amount of (semi-) constructive information is still extractable from proofs using Σ^0_2-DNE.

From the proof of Lemma 4.17 we get that Σ^0_2-LLPO is not even limit realizable. For such a realizer would, by the Limit Lemma (cf. [48, 15.4] and [45, Theorem 4]) and Post's Theorem (cf. [48, 14.9]), be recursive in Π^0_1. Therefore it would be recursive in the ε-function g. But this leads to the contradiction also found in the proof of Lemma 4.17.

Chapter 5

Overview of Related Work

Our calibration project resembles the main question of reverse mathematics:

> Which set existence axioms are needed to prove the theorems of ordinary, non-set-theoretic mathematics?[1]

The reason for the resemblance is in essence the word "needed". For reverse mathematics investigates theorems from mathematical practice and certain systems of set-theoretic axioms. It turns out that there is a small number of such systems, each corresponding to a subclass of the considered mathematical theorems such that, for proving theorems in a certain subclass, axioms from the corresponding subclass are *needed*. Furthermore, these axiom systems are sufficient in many cases.

Below we present some of the results of reverse mathematics that are of interest to this project. We then introduce a refined reverse mathematics which makes proof theoretically motivated distinctions between mathematical theorems identified by ordinary reverse mathematics. Next, an essentially different calibration view-point from constructive mathematics is considered.

5.1 Classical Reverse Mathematics

The term *reverse mathematics* is used for a metamathematical branch which has a reversed approach; the theorems of mathematical practice become the hypotheses and the axioms are the conclusions. Sometimes the term also figures as synonymous with the monograph [49] by Simpson. We follow this trend and coin the phrase *classical reverse mathematics* for a broader area — hoping this not to be a cause of confusion.

[1] [49, p. 2].

5.1.1 Reverse Mathematics

The title of [49] sums up its setting, *Subsystems of Second Order Arithmetic.* Second order arithmetic is two-sorted arithmetic with both quantifiers for numbers and sets of numbers. This corresponds to type 0 respectively type 1 quantifiers, by relating a set with its characteristic function. Thus, the context is strong enough to reason about real numbers and continuous functions.

One of the discoveries of reverse mathematics is that very few subsystems are needed to precisely characterise large parts of ordinary mathematics; the most important among those are RCA_0, ACA_0 and WKL_0.

RCA_0 is the weakest system we shall look at. It has basic axioms for addition, multiplication and ordering on the natural numbers. Induction restricted to Σ^0_1-formulas and comprehension for Δ^0_1-formulas is included. No parameter restrictions are made on the formula classes, ie. they are allowed to have both number and set variables.

From basic recursion theory, we know that the Δ^0_1 sets are exactly the recursive ones (cf. [48, 14.6]). Using this fact one finds that the recursive sets form the smallest ω-model (ie. a model whose ground type is modelled by ω — the set of natural numbers) of RCA_0. For this reason, RCA_0 is in [49] promoted as corresponding to the "foundational program" of constructivism.[2]

In RCA_0 many of the basic properties of the real numbers and closure properties of the class of continuous functions can be established; even the intermediate value theorem[3] is provable in RCA_0.[4]

ACA_0 is obtained from RCA_0 by adding comprehension for arithmetical formulas. In the same spirit as with RCA_0, one finds that the arithmetical sets form the smallest ω-model of ACA_0. Already in ACA_0 a significant part of mathematics can be proved, and, for the reverse direction, important mathematical principles are equivalent to ACA_0 over RCA_0. For instance, the principle of monotone convergence[5] (PCM), the Bolzano-Weierstraß principle[6] (BW) and the limit superior principle[7] (Limsup) are all equivalent to ACA_0 (cf. [49, III.2.2]).

[2] Cautious remarks to this correspondence are made in [49, I.8.9].

[3] For every continuous function $f : [0,1] \to \mathbb{R}$ with $f(0) \leq 0 \wedge f(1) \geq 0$ there exists a point $x \in [0,1]$ such that $f(x) = 0$.

[4] The proof of this last statement is sketched and discussed in Sect. 8.3.

[5] Every bounded increasing sequence of rationals is convergent.

[6] Every bounded sequence of rationals has a limit point.

[7] Every bounded sequence of rational has a largest limit point.

WKL_0. A group of principles concerning continuous functions are provable in ACA_0 and not in RCA_0. But on the other hand they do not entail arithmetical comprehension; eg. the attainment of the maximum principle[8] and the Heine-Borel covering lemma[9]. In turns out that a comprehension principle known as weak König's lemma captures these intermediate theorems of classical analysis — along with many other non-constructive theorems.

Weak König's lemma states that any infinite binary tree has an infinite branch. Adding this principle to RCA_0 defines the system WKL_0. This results in a strictly stronger system since Kleene's singular tree is a recursive infinite binary tree with no recursive infinite branch (see [3]). The construction of the singular tree uses two recursively enumerable and recursively inseparable sets, as does the proof that Σ^0_1-LLPO is not provable in HA (Lemma 4.13).

The relationship between weak König's lemma and Σ^0_1-LLPO is closer yet; actually they are equivalent over $\mathsf{HA}^\omega + \mathsf{AC}^{0,0}$ (cf. [20] and [33]). Chap. 8 will include a discussion of the kinship between WKL_0 and $\mathsf{HA}^\omega+\mathsf{AC}^{0,0}+\Sigma^0_1$-$\mathsf{LLPO}$.

5.1.2 A Refinement

For proof mining purposes the characterisations above can be somewhat crude. If a proof only uses, say the Bolzano-Weierstraß principle, *without* function parameters (respectively with function parameters kept fixed) — which indeed is the case for many applications — then much more information can be extracted from the proof, than if we merely exploit the fact that we *a fortiori* have used the unrestricted principle. Such a refined analysis has been carried out in [28] and [32] for some principles of mathematical analysis. In a sense, this deviates from the main question of [49]; the focus is no longer on set-theoretic axioms used to prove the full second order version of certain principles, but on the use of instances of these principles.

When restricting PCM, one finds that it only implies comprehension for Σ^0_1 formulas (without function parameters),[10] whereas the unrestricted form implies full arithmetical comprehension. In terms of provably total functions, the borderline is found to be between BW and Limsup. BW (and hence PCM) in a weak[11] second-order context corresponds to $\mathsf{PRA}+\Sigma^0_1$-IA whereas Limsup corresponds to $\mathsf{PRA} + \Sigma^0_2$-$\mathsf{IA}$. The former system only proves the totality of the primitive recursive functions, whereas the latter also proves for instance

[8] Every continuous function on $[0,1]$ attains its maximum.

[9] Every covering of $[0,1]$ by a sequence of open sets has a finite subcovering.

[10] Cf. [32, 5.7].

[11] Weaker than RCA_0 since Σ^0_1-CA (without set/function parameters) would in the presence of Σ^0_1-IA^+ (Σ^0_1 induction with set/function parameters) allow to derive Σ^0_2-IA (without set/function variables) and hence the totality of the Ackermann function.

the Ackermann function to be total.

5.2 Constructive Reverse Mathematics

The main question of constructive reverse mathematics could be phrased: "what amount of classical logic is needed to prove the theorems of ordinary mathematics". On the other hand, countable choice ($\mathsf{AC}^{0,0}$) is included in the context. There is no established program of constructive reverse mathematics, but instead a series of isolated results can be gathered by their answers to the question given above.

The subject dates back to Brouwer and the so-called Brouwerian counterexamples. They are the subject of the next section and can be viewed as the "reverse" part of a constructive calibration program. Following this, we discuss a very recent survey article whose subject is constructive reverse mathematics.

5.2.1 Brouwerian Counterexamples

One easily gets the impression that Brouwer's constructive approach is mainly destructive. Intuitionistic reasoning disallows the LEM and therefore most of, for instance and in particular, mathematical analysis is no longer valid in intuitionistically based mathematics. This impression turns out to be oversimplified — in a sense, intuitionistic logic can be viewed as a refinement of classical logic (eg. Sect. 3.1.1) and intuitionistic mathematics and classical mathematics are incomparable (eg. non-classical axioms on choice sequences can be consistently added. See also Sect. 8.3). Given these remarks, we now present some negative (destructive) results concerning analysis.

Brouwerian counterexamples were first used by Brouwer to demonstrate how certain mathematical principles were inherently classical. This is obtained by showing that the principle in question implies a principle of classical (and not intuitionistic) logic or, in the earlier cases, a statement that in a constructive interpretation gives a solution to some unsolved problem, like the Riemann hypothesis. Later the same technique was used in Bishop's constructive mathematics, but with a more systematic approach, not directly using unsolved problems. A group of ineffective logical principles — Bishop coined them *Omniscience Principles*[12] — were made for this purpose. They are LPO, WLPO and LLPO (see Sect. 2.3); also Markov's principle and a so-called weak Markov's principle are considered, but counterexamples based on

[12]Mandelkern gives an informal introduction to the area and its history in [43]. He explains, "The term 'omniscience' is used [. . .] to remind us that we are not omniscient!"

those principles are, so to speak, not first-class counterexamples as indicated in for instance [43, p. 20].

The counterexamples can be seen as the reverse part of a reverse mathematics in an intuitionistic context. We shall see that some of these counterexamples can be used to prove one direction in the calibrations we perform in the following chapters.

A *Brouwer*ian counterexamples sketched in [5] shows that the intermediate value theorem implies LLPO in a constructive context. As we saw in connection with Definition 2.12, LLPO is closely connected to Σ^0_1-LLPO, which in turn corresponds to the second-order theory WKL_0 of reverse mathematics. This already suggests that classical and constructive reverse mathematics do not agree on the status of the intermediate value theorem, since it is provable in RCA_0. Sect. 8.3 will discuss this in detail.

5.2.2 Precise Calibrations of Entailed Classical Logic

To make it easier to come up with Brouwerian counterexamples,[13] Mandelkern in [42] found a series of principles all equivalent to LPO — Bolzano-Weierstraß for example. This led to further research whose aim was, given a theorem of classical mathematics T, to find a logical principle P such that not only $T \Rightarrow P$, but also $P \Rightarrow T$ — where the implications are proved in Bishop's constructive mathematics.

Very recently a survey of this research has been given in [21]. As the title of the paper suggests, this may be viewed as work in constructive reverse mathematics. The paper considers both the omniscience principles and a group of very weak principles resembling Markov's principle.

Using the terms that have been introduced in the present chapter, the project of this thesis, in particular the following chapters, is to study *refined constructive reverse mathematics*.

[13] [43, p. 19] gives this motivation.

Chapter 6

The Strength of Σ^0_1-LEM and Π^0_1-LEM

This chapter gives the first calibration results. They are concerned with Σ^0_1-LEM and Π^0_1-LEM. We start out with a rather detailed study of Σ^0_1-LEM resulting in enough results to close up the investigation for our purpose. Following this, we examine Π^0_1-LEM. This turns out to leave behind an equivalence between Π^0_1-LEM and an analytic supremum principle and an open problem.

6.1 Introducing Principles from Analysis

Definition 6.1. Let f be a function of type 1 representing a sequence of rationals via the coding from Sect. 2.4.1. We define the principle of convergence for bounded, monotone sequences as:

$$\begin{aligned}\mathsf{PCM}(f) :\equiv & \forall n \,(0 \leq_{\mathbb{Q}} f(n+1) \leq_{\mathbb{Q}} f(n)) \\ & \rightarrow \exists g^1 \forall k \forall m \left(|f(g(k)) -_{\mathbb{Q}} f(g(k)+m)|_{\mathbb{Q}} <_{\mathbb{Q}} 2^{-k} \right) \quad .\end{aligned}$$

Which immediately gives — and using $\mathsf{AC}^{0,0}$ follows from — its arithmetical formulation:

$$\begin{aligned}\mathsf{PCM}_{\mathrm{ar}}(f) :\equiv & \forall n \,(0 \leq_{\mathbb{Q}} f(n+1) \leq_{\mathbb{Q}} f(n)) \\ & \rightarrow \forall k \exists n \forall m \left(|f(n) -_{\mathbb{Q}} f(n+m)|_{\mathbb{Q}} <_{\mathbb{Q}} 2^{-k} \right) \quad .\end{aligned}$$

Remark 6.2. The restriction to non-increasing sequences and the particular lower bound 0 is inessential.

An instance of an analytic principle is given by a term t of HA. As with the logical principles defined in Sect. 2.3, we allow number parameters in instances of PCM^- and $\mathsf{PCM}^-_{\mathrm{ar}}$, but not function parameters. To remind ourselves and others of this, we add a $^-$ to the schema name when referring to this restricted form of it. The same convention also applies to other mathematical principles that will be defined in the next chapters.

Clearly, the existence of a limit with a modulus of convergence γ,

$$\exists x \in \mathbb{R} \forall k \forall m \geq \gamma(k)(|x -_{\mathbb{R}} f(m)| <_{\mathbb{R}} 2^{-k}) \ ,$$

implies PCM^- (and hence $\mathsf{PCM}^-_{\mathrm{ar}}$) over HA^ω. But $\mathsf{PCM}^-_{\mathrm{ar}}$ does not allow one to derive existence of the limit, since by our definition, a real number must have fixed rate of convergence 2^{-n}, and this is only implicitly given in $\mathsf{PCM}^-_{\mathrm{ar}}$. With the use of $\mathsf{AC}^{0,0}$ however, one can over HA^ω get PCM^- which in turn implies the existence of the limit with the modulus of convergence. Actually, only $\mathsf{QF}\text{-}\mathsf{AC}$ is needed to get PCM^- from just the existence of a limit. This is due to the fact that the existence of a limit,

$$\exists x \in \mathbb{R} \forall k \exists n \forall m \geq n(|x -_{\mathbb{R}} f(m)| <_{\mathbb{R}} 2^{-k}) \ ,$$

is equivalent to

$$\exists x \in \mathbb{R} \forall k \exists n > k(|x -_{\mathbb{R}} f(n)| <_{\mathbb{R}} 2^{-k}) \ ,$$

when f is monotone.

From classical analysis it is a well-known fact that a real-valued function defined on a closed interval is of bounded variation if and only if it is the difference of two increasing functions. We formulate one direction of the restriction of this theorem to the uniformly continuous functions in the following principle.

Definition 6.3. Define $\mathsf{PBV}(f)$ as

$$\begin{aligned} f \in C[0,1] \to \big(&\exists b \in \mathbb{Q} \forall \underline{r}^0 (V_{\underline{r}}(f) \leq_{\mathbb{Q}} b) \to \exists g^{1\to 1}, h^{1\to 1} \forall x, y \in_{\mathbb{R}} [0,1] \\ &[x \leq_{\mathbb{R}} y \to (g(x) \leq_{\mathbb{R}} g(y) \wedge h(x) \leq_{\mathbb{R}} h(y)) \wedge f(x) =_{\mathbb{R}} g(x) - h(x)]\big) \ , \end{aligned}$$

where

$$V_{r_1,\ldots,r_m}(f) :\equiv \sum_{i=0}^{m-1} |f(r_{i+1}) - f(r_i)| \, .$$

Remark 6.4. By primitive recursive coding of all finite sequences the $\forall \underline{r}^0$ quantifier is a type 0 quantifier.

We shall use a result due to N.D. Goodman.

Theorem 6.5 (Goodman's theorem). $\mathsf{HA}^\omega + \mathsf{AC}$ *is conservative over* HA*, that is: If* A *is a formula of* HA *then*

$$\mathsf{HA}^\omega + \mathsf{AC} \vdash A \Rightarrow \mathsf{HA} \vdash A \ .$$

6.2 Monotone Convergence and Σ^0_1-LEM

The following proposition states that $\mathsf{PCM}^-_{\mathrm{ar}}$ is at least as strong as Σ^0_1-LEM. The proof uses a construction also presented in [56, 5.4.4]. There however, it is used to establish a Brouwerian counterexample to PCM based on LPO. Following [56] we use an increasing sequence bounded from above by 2 — this is of course inessential.

Proposition 6.6. $\mathsf{HA} + \mathsf{PCM}^-_{\mathrm{ar}} \vdash \Sigma^0_1\text{-}\mathsf{LEM}$.

Proof. Let the Σ^0_1-LEM instance be given by a term $t[\underline{a}]$ such that we aim to decide $\exists n(t(n,\underline{a}) = 0) \vee \neg\exists n(t(n,\underline{a}) = 0)$. Define

$$f(m,\underline{a}) = \begin{cases} 1 - 2^{-m} & \text{if } \neg\exists n \le m\,(t(n,\underline{a}) = 0) \\ 2 - 2^{-m} & \text{if } \exists n \le m\,(t(n,\underline{a}) = 0) \end{cases} .$$

Clearly, f is monotone increasing (in its first argument) and $f(m,\underline{a}) \in [0,2]$. By $\mathsf{PCM}^-_{\mathrm{ar}}$ we therefore get $\exists n_0 \forall m \left(f(n_0+m,\underline{a}) - f(n_0,\underline{a}) < \frac{1}{4}\right)$. Since f only takes rational values we can prove $f(n,\underline{a}) \le 1 \vee f(n,\underline{a}) > 1$. For each case we have:

(1) $f(n_0,\underline{a}) \le 1$: Using the definition of n_0 we get

$$\forall m \ge n_0 \left(f(m,\underline{a}) < \tfrac{5}{4}\right) . \tag{6.1}$$

But we also have

$$\begin{aligned} \exists n(t(n,\underline{a}) = 0) &\to \exists n \forall m \ge n \left(f(m,\underline{a}) = 2 - 2^{-m}\right) \\ &\to \exists n \forall m \ge \max\{n,1\} \left(f(m,\underline{a}) \ge \tfrac{6}{4}\right) , \end{aligned}$$

which together with (6.1) yields $\neg\exists n(t(n,\underline{a}) = 0)$.

(2) $f(n_0,\underline{a}) > 1 \to \exists n \le n_0(t(n,\underline{a}) = 0) \to \exists n(t(n,\underline{a}) = 0)$

That is, from $f(n_0,\underline{a}) \le 1 \vee f(n_0,\underline{a}) > 1$ we get the desired Σ^0_1-LEM. ☐

Remark 6.7 *(Strengthening).* As a corollary to the proof, we find the following, somewhat stronger, result: There is a term Φ^2 of HA^ω restricted to only have the recursor constant R_0 for type 0 such that

$$\mathsf{HA}^\omega \vdash \forall f^1(\mathsf{PCM}_{\mathrm{ar}}(\Phi(f)) \to \Sigma^0_1\text{-}\mathsf{LEM}(f)) .$$

When f is a primitive recursive function, the fact that Φ at most uses R_0 ensures that $\Phi(f)$ is again primitive recursive (cf. [24]).

That $\mathsf{PCM}^-_{\mathrm{ar}}$ is in fact no stronger than Σ^0_1-LEM is expressed in the following theorem. Its proof is based on proof interpretations. The idea is that since $\mathsf{PCM}^-_{\mathrm{ar}}$ is easily proved in PA, we know that HA proves a version of $\mathsf{PCM}^-_{\mathrm{ar}}$ formulated sufficiently negatively. The main task is then to show that Σ^0_1-LEM is enough to get the original formulation back.

Theorem 6.8. $\mathsf{HA} + \Sigma^0_1\text{-}\mathsf{LEM} \vdash \mathsf{PCM}^-_{\mathrm{ar}}$.

Proof. We consider a classical proof of $\mathsf{PCM}^-_{\mathrm{ar}}$:

$$\begin{aligned}\mathsf{PA} \vdash \forall \underline{a}\big[&\forall n\,(0 \leq t(n+1,\underline{a}) \leq t(n,\underline{a})) \\ &\to \forall k \exists n \forall m\, \big(|t(n,\underline{a}) - t(n+m,\underline{a})| < 2^{-k}\big)\,\big] \ .\end{aligned}$$

Define $\tilde{f}(n,\underline{a}) := \max\{0, \min_{i\leq n}\{f(i,\underline{a})\}\}$. $\tilde{f}$ is decreasing, non-negative, and if already f were so, then it is equal to f. Therefore it is enough to consider $\tilde{t}$. That is,

$$\mathsf{PA} \vdash \forall k \exists n \forall m\, \Big(\big|\tilde{t}(n,\underline{a}) - \tilde{t}(n+m,\underline{a})\big| < 2^{-k}\Big) \ .$$

Using negative translation and functional interpretation (cf. Example 3.11) we get a term Φ, such that

$$\mathsf{HA}^\omega \vdash \forall k \forall g^1\, \Big(\big|\tilde{t}\,(\Phi(k,g,\underline{a}),\underline{a}) - \tilde{t}\big(\Phi(k,g,\underline{a}) + g(\Phi(k,g,\underline{a}))\big)\big| < 2^{-k}\Big) \ .$$

Let $A_0(m,n,k,\underline{a}) :\equiv \big|\tilde{t}(n,\underline{a}) - \tilde{t}(n+m,\underline{a})\big| < 2^{-k}$. To finish the proof we need to show, in some appropriate setting, that

$$\forall g A_0\big(g(\Phi(k,g,\underline{a})), \Phi(k,g,\underline{a}), k, \underline{a}\big) \to \exists n \forall m A_0(m,n,k,\underline{a}) \ ,$$

where appropriate means that it is conservative over $\mathsf{HA} + \Sigma^0_1$-LEM.

We have

$$\begin{aligned}&\forall n(\exists m \neg A_0(m,n,k,\underline{a}) \to \exists l \neg A_0(l,n,k,\underline{a})) \overset{\Sigma^0_1\text{-}\mathsf{LEM}}{\to} \\ &\forall n \exists l(\exists m \neg A_0(m,n,k,\underline{a}) \to \neg A_0(l,n,k,\underline{a})) \overset{\mathsf{AC}^{0,0}}{\to} \\ &\exists g_0 \forall n(\exists m \neg A_0(m,n,k,\underline{a}) \to \neg A_0(g_0(n),n,k,\underline{a})) \ .\end{aligned}$$

Our claim is that $\Phi(k,g_0,\underline{a})$ is the n we are looking for — that is, we need to show $\forall m A_0(m,\Phi(k,g_0,\underline{a}),k,\underline{a})$.

By assumption we have

$$A_0\big(g_0(\Phi(k,g_0,\underline{a})), \Phi(k,g_0,\underline{a}), k, \underline{a}\big) \ . \tag{6.2}$$

Now for all m we find that

$$\neg A_0(m, \Phi(k, g_0, \underline{a}), k, \underline{a}) \to \neg A_0\big(g_0(\Phi(k, g_0, \underline{a})), \Phi(k, g_0, \underline{a}), k, \underline{a}\big),$$

which contradicts (6.2), and we get $\neg\neg A_0(m, \Phi(k, g_0, \underline{a}), k, \underline{a})$, and then $A_0(m, \Phi(k, g_0, \underline{a}), k, \underline{a})$. So finally we have $\forall m A_0(m, \Phi(k, g_0, \underline{a}), k, \underline{a})$.

That is,

$$\mathsf{HA}^\omega + \mathsf{AC} + \Sigma^0_1\text{-}\mathsf{LEM} \vdash \forall k \exists n \forall m \left(\left|\tilde{t}(n, \underline{a}) - \tilde{t}(n+m, \underline{a})\right| < 2^{-k}\right) \quad ,$$

where we remark that instances of Σ^0_1-LEM are terms of HA only. The deduction theorem and Goodman's theorem now finish the proof. □

From the two results above we gather the following corollary, which states analogous results for un-arithmetical PCM^-.

Corollary 6.9.

$$\mathsf{HA}^\omega + \mathsf{AC}^{0,0} + \Sigma^0_1\text{-}\mathsf{LEM} \vdash \mathsf{PCM}^- \quad ,$$

and

$$\mathsf{HA}^\omega + \mathsf{PCM}^- \vdash \Sigma^0_1\text{-}\mathsf{LEM} \quad .$$

6.3 Discussion and Improvement

A natural question now arises. Is Goodman's theorem really needed? The result simply states that the equivalence can be proved in HA, so why not do this instead of just, via some rather technical theorem, showing that it is possible. A first attempt would be to examine the classical proof of $\mathsf{PCM}^-_{\mathrm{ar}}$ and check whether any classical reasoning beyond Σ^0_1-LEM is used. Unfortunately, the standard proof of $\mathsf{PCM}^-_{\mathrm{ar}}$ is by contradiction:

Let $\{a_n\}$ be a bounded, decreasing sequence. Assume for contradiction that it is not Cauchy. Then there exists a k_0 such that for all numbers a_n in the sequence we can always find one with higher index that is farther from a_n than 2^{-k_0}. This way we reach the lower bound of the sequence in finitely many steps, which is a contradiction.

That is, we have proved $\forall k \neg\neg\exists n \forall m \left(|a_n - a_{n+m}| < 2^{-k}\right)$.[1] It therefore seems that we need a whole sequence of Σ^0_2-DNE instances to prove $\mathsf{PCM}^-_{\mathrm{ar}}$. Since we know that Σ^0_1-LEM is weaker than Σ^0_2-DNE (Theorem 4.12) this is not satisfactory. Instead we have the following:

[1] In [17] p. 110 this is proved formally. The name *non-oscillating* is used for a sequence satisfying $\forall k \neg\neg\exists n \forall m \left(|a_n - a_{n+m}| < 2^{-k}\right)$ — the negative translation of the Cauchy criterion.

Alternative proof of Theorem 6.8. By scaling the sequence we may assume, without loss of generality, that 1 is an upper bound. By induction on k we show that

$$\forall k \exists i \in \{1, \ldots, 2^k\} \exists n \forall m \left(\frac{i-1}{2^k} \leq \tilde{t}(n+m) \leq \frac{i}{2^k} \right) \quad . \tag{6.3}$$

The base case is trivial.
Induction step:
Assume that

$$\exists i \in \{1, \ldots, 2^k\} \exists n \forall m \left(\frac{i-1}{2^k} \leq \tilde{t}(n+m) \leq \frac{i}{2^k} \right) \quad .$$

With Σ_1^0-LEM we can prove

$$\exists n \left(\tilde{t}(n) < \frac{2i-1}{2^{k+1}} \right) \vee \neg \exists n \left(\tilde{t}(n) < \frac{2i-1}{2^{k+1}} \right) \quad .$$

In the first case, using the induction hypothesis and the fact that $\tilde{t}$ represents a decreasing function, we get

$$\exists n \forall m \left(\frac{(2i-1)-1}{2^{k+1}} \leq \tilde{t}(n+m) \leq \frac{2i-1}{2^{k+1}} \right) \quad .$$

And therefore

$$\exists j \in \{1, \ldots, 2^{k+1}\} \exists n \forall m \left(\frac{j-1}{2^{k+1}} \leq \tilde{t}(n+m) \leq \frac{j}{2^{k+1}} \right) \quad ,$$

with for instance $j = 2i - 1$.
In the second case we have

$$\exists n \forall m \left(\frac{2i-1}{2^{k+1}} \leq \tilde{t}(n+m) \leq \frac{2i}{2^{k+1}} \right) \quad ,$$

by the induction hypothesis. And so, with $j = 2^{k+1}$, we get the desired result.

From (6.3) Theorem 6.8 follows immediately. □

Remark 6.10 *(Strengthening).* From the proof it follows that there is a term Φ of HA^ω using at most R_0 with type $1 \to (0 \to 1)$ such that

$$\mathsf{HA}^\omega \vdash \forall f^1 \left(\forall n \Sigma_1^0\text{-}\mathsf{LEM}\big((\Phi(f))(n)\big) \to \mathsf{PCM}_{\mathrm{ar}}(f) \right) \quad .$$

So, Goodman's theorem was not necessary in the first place. We have included it to illustrate a somewhat pragmatic point. The development presented here was also the actual development of the proof. From the first proof which used sophisticated proof theoretic techniques but with a straightforward idea, to the second proof which used no metamathematics but a rather complex instance of induction.

[31] (and Goodman's own proof) points to a way of avoiding the use of AC, for the paper shows how Goodman's conservation result does not extend to subsystems of HA, where the induction axiom is restricted to Σ_n^0 formulas. This suggests that a proof by induction could work, and also that it might be a complex instance — which was how the second proof was found.

Therefore, one way of using Goodman's theorem is to first make free use of the axiom of choice and higher types, and then do a new proof exploiting unrestricted induction. That this might be a gain, is suggested by results from reverse mathematics indicating that actual mathematical practice does not need complex induction instances — such instances are not part of natural reasoning, so to speak.

Remarks 6.7 and 6.10 show that a whole sequence of Σ_1^0-LEM instances is needed to get one $\mathsf{PCM}_{\mathrm{ar}}^-$ instance, but one instance of $\mathsf{PCM}_{\mathrm{ar}}^-$ is enough to get one instance of Σ_1^0-LEM. One might ask if our proof of Theorem 6.8 is too crude — the equivalence could be even on the level of number parameter-free instances. However, the following proposition answers the question negatively.

Proposition 6.11. *There exists a closed term t of HA such that for all closed terms t' of HA*

$$\mathsf{HA} \nvdash \Sigma_1^0\text{-}\mathsf{LEM}(t') \to \mathsf{PCM}_{\mathrm{ar}}^-(t) \ .$$

Proof. Assume that

$$\mathsf{HA} \vdash \Sigma_1^0\text{-}\mathsf{LEM}(t') \to \mathsf{PCM}_{\mathrm{ar}}^-(t) \ .$$

Then

$$\mathsf{HA} \vdash \exists l(t'(l) = 0) \to \mathsf{PCM}_{\mathrm{ar}}^-(t) \ ,$$

and

$$\mathsf{HA} \vdash \neg\exists l(t'(l) = 0) \to \mathsf{PCM}_{\mathrm{ar}}^-(t) \ .$$

Assuming t to represent a monotone and bounded sequence, modified realizability gives us two terms φ, ψ representing recursive functions such that

$$\neg\exists l(t'(l) = 0) \to \forall k, m \left(|t(\varphi(k)) - t(\varphi(k) + m)| < 2^{-k}\right) \ ,$$

and

$$t'(l) = 0 \rightarrow \forall k, m \left(|t(\psi(l,k)) - t(\psi(l,k)+m)| < 2^{-k} \right) \ ,$$

both hold.

In either case we find that t has a recursive rate of convergence — φ or $\lambda k.\psi(l_0, k)$ for some l_0.[2] We now present a primitive recursive function from [50] for which this is not the case:[3]

Define a primitive recursive function by

$$g(m,n) = \begin{cases} 0 & \text{if } \forall l \leq n \neg T(m,m,l) \\ 1 & \text{otherwise.} \end{cases}$$

And let a sequence be given by

$$f(n) = \sum_{k=0}^{n} g(k,n) \cdot 2^{-k} \ .$$

Now, f is primitive recursive, monotone and bounded. Assume that it has recursive rate of convergence b, ie.

$$\forall k (|f(b(m)+k) - f(b(m))| < 2^{-m}) \ .$$

We may choose b such that $b(m) \geq m$. For $n \geq b(m)$ we have

$$2^{-m} > \sum_{k=0}^{n} g(k,n) \cdot 2^{-k} - \sum_{k=0}^{b(m)} g(k,b(m)) \cdot 2^{-k} \geq \sum_{k=0}^{b(m)} (g(k,n) - g(k,b(m))) 2^{-k} \ .$$

Since $0 \leq g(k,n) - g(k,b(m)) \leq 1$ for $n \geq b(m)$, looking at the m'th term of the sum gives

$$g(m,n) = g(m,b(m)) \quad \text{for } n \geq b(m) \, .$$

Hence, $\exists l T(m,m,l)$ if and only if $\exists l \leq b(m) T(m,m,l)$, which is decidable. This contradicts the undecidability of the halting problem.

□

Generally speaking, the proposition argues that if we made an even more refined analysis, $\mathsf{PCM}^-_{\mathrm{ar}}$ and Σ^0_1-LEM would not be equivalent. But note also that for our purpose, there is no reason to carry out this refinement. As the results of this and the following chapters clearly indicate, an analysis

[2] Note that classical reasoning on the meta-level gives us insight in what can be proved intuitionistically.

[3] A bounded, monotone, recursive sequence without recursive rate of convergence is called a Specker sequence, referring to [50].

on the level of parameter-free instances would hardly leave any equivalences behind between analytic and logical principles, and it would therefore require a different and whole new set of motivations.

Consequently, the following results shall not address the question on whether the equivalences are on the level of number parameter-free instances or not.

6.4 A Weak Supremum Property and Π^0_1-LEM

We now turn to the treatment of Π^0_1-LEM. The section consists of refinement of results from [47]. To be faithful to the original proofs we note that *without loss of generality we may assume that an instance given by f of* Π^0_1-LEM *is binary and has at most one n such that $f(n) = 1$.*

Theorem 6.12.

$$\mathsf{HA} + \mathsf{PBV}^- \vdash \Pi^0_1\text{-}\mathsf{LEM} \ .$$

Proof. Consider an instance of Π^0_1-LEM given by a binary sequence $\{a_n\}$ such that

$$\forall n \forall m > 0(a_n = 1 \to a_{n+m} = 0) \ .$$

For each $n \geq 1$ we can construct a uniformly continuous function $g_n : [0,1] \to \mathbb{R}$ such that

$$\exists \bar{r}(V_{\bar{r}}(g_n) = 1) \ ,$$

$$\forall \bar{r}(V_{\bar{r}}(g_n) \leq 1) \ ,$$

$$p \in_{\mathbb{Q}} [0,1] \to g_n(p) \in_{\mathbb{R}} [0, 2^{-n}] \ ,$$

and

$$q \notin_{\mathbb{Q}} [0, 2^{-n}] \to g_n(q) = 0 \ .$$

Simply make a zigzag graph by connecting 2^n+1 points in $[0, 2^{-n}]$ alternating between 0 and 2^{-n} with line segments, taking 0 as the first and 2^{-n} as the last point. Outside $[0, 2^{-n}]$ we put g_n to 0. The four requirements above can be verified in a straightforward manner.

For each x the sequence $\sum_{n=1}^{m} a_n g_n(x)$ is a Cauchy sequence (with rate 2^{-n}) and we let $f(x)$ be the real number thus defined. The moduli of uniform continuity for the g_n's can be chosen to have the form $\omega_{g_n}(k) := k + l(n)$, for some primitive recursive l[1]. Since

$$|f(x) - f(y)| \leq 2^{-n+2} + |g_n(x) - g_n(y)| \ ,$$

$\omega(k) := k + 1 + l(k+3)$ is a modulus of uniform continuity for f. Now the variation of f is bounded by 1, for assume that for some $\bar{r}$, $V_{\bar{r}}(f) > 1$. Then by choosing n sufficiently large also $V_{\bar{r}}(g_n) > 1$, which is a contradiction. So by PBV^- there exists two increasing functions k, h such that $f = h - k$. Define the increasing function $\lambda := h + k$. Without loss of generality we assume that $\lambda(0) = 0$. It is easily seen that the variation of f on any interval is bounded by that of λ. The sequence $\sum_{n=1}^{m} a_n 2^{-n}$ is Cauchy with rate 2^{-n}. Let s be its limit.

Now if $\exists n(a_n = 1)$, then $s = 2^{-n}$ and $f = g_n$. The variation of $f = g_n$ on $[0, 2^{-n}]$ is bounded by that of λ, therefore $\lambda(s) \leq 1$. That is, $\lambda(s) < 1 \to \forall n(a_n = 0)$. Also $\lambda(s) > 0$ means that $s \neq 0$ hence $\neg\forall n(a_n = 0)$. ☐

Remark 6.13 *(Strengthening).* For a term Φ of HA^ω using at most R_0, representing a type 2 functional, we have

$$\mathsf{HA}^\omega \vdash \forall f^1 \left(\mathsf{PBV}(\Phi(f)) \to \Pi^0_1\text{-}\mathsf{LEM}(f)\right) \ .$$

The other direction does not come as easily. In fact, the precise strength of PBV^- is not known. PBV is proved to be equivalent to WLPO in [47], via a definition of *weak supremum*. The first part of the original proof has a counterpart in the next lemma.

Lemma 6.14.

$$\mathsf{HA}^\omega + \mathsf{AC}^{0,0} + \Pi^0_1\text{-}\mathsf{LEM} \vdash \mathsf{WSup}^- \ ,$$

where we for a sequence of real numbers in $[0, 1]$ *given by a function* $f^{0\to 1}$ *define the weak supremum principle,* $\mathsf{WSup}^-(f)$*, by*

$$\forall n(f(n) \in_{\mathbb{R}} [0, 1]) \to \exists x^1 \forall y^1 (\forall n^0 (y \geq_{\mathbb{R}} f(n)) \leftrightarrow y \geq_{\mathbb{R}} x) \ .$$

We denote this x by $\mathrm{wsup}_n f(n)$*.*

Remark 6.15. It follows from the definition that $\forall n(\mathrm{wsup}_m f(m) \geq_{\mathbb{R}} f(n))$.

Proof. By a bisection argument using Π^0_1-LEM and $\mathsf{AC}^{0,0}$ we can define a Cauchy sequence $x(\cdot)$ with rate 2^{-n} such that[4]

$$\forall m[\forall n(f(n) \leq_{\mathbb{R}} x(m+1) +_{\mathbb{Q}} 2^{-m}) \wedge \neg\forall n(f(n) \leq_{\mathbb{R}} x(m+1))] \ . \tag{6.4}$$

Let x^1 be the real number defined by $x(\cdot)$. From this it follows that $\forall n(x \geq_{\mathbb{R}} f(n))$. For assume $f(n) >_{\mathbb{R}} x$, then using Lemma 2.20 there would be an m such that $x(m+1) +_{\mathbb{Q}} 2^{-m} <_{\mathbb{R}} f(n)$, which contradicts (6.4). Now we easily get $y \geq_{\mathbb{R}} x \to \forall n(y \geq_{\mathbb{R}} f(n))$.

The other direction follows similarly. ☐

[4]Refer to the proofs of the Proposition 8.4 and 8.14 for details of the bisection approach.

Remark 6.16 *(Strengthening).* For a term Φ of HA^ω of type $1 \to (0 \to 1)$ using at most R_0 we have

$$\mathsf{HA}^\omega \vdash \forall f^1 \left(\forall n \Pi^0_1\text{-}\mathsf{LEM}((\Phi(f))(n)) \to \mathsf{WSup}(f)\right) \ .$$

The weakness of WSup^- from a constructive viewpoint lies in the fact that we do not explicitely require that there exists numbers arbitrarily close to the least upper bound. To have this too — the full least upper bound principle — would entail Σ^0_1-LEM. This follows directly from the construction in Proposition 6.6.

This way we also show that WSup^- entails Π^0_1-LEM:

Proposition 6.17.

$$\mathsf{HA}^\omega + \mathsf{WSup}^- \vdash \Pi^0_1\text{-}\mathsf{LEM} \ .$$

Proof. We use the construction from the proof of Proposition 6.6. Define

$$f(m) := \begin{cases} 1 - 2^{-m} & \text{if } \neg \exists n \leq m\, (t(n) \neq 0) \\ 2 - 2^{-m} & \text{if } \exists n \leq m\, (t(n) \neq 0) \ , \end{cases}$$

and let $x := \mathrm{wsup}(f)$. We can prove $x \leq_{\mathbb{R}} \frac{6}{4} \lor x \geq_{\mathbb{R}} \frac{5}{4}$. In the first case, we get a contradiction to $\exists n(t(n) \neq 0)$, hence $x \leq_{\mathbb{R}} \frac{6}{4} \to \forall n(t(n) = 0)$. Since $\forall n(t(n) = 0)$ implies $\mathrm{wsup}(f) \leq 1$, the other case contradicts $\forall n(t(n) = 0)$, ie. $x \geq_{\mathbb{R}} \frac{5}{4} \to \neg \forall n(t(n) = 0)$. $\square$

Remark 6.18 *(Strengthening).* For a term Φ of HA^ω of type 2 using at most R_0 we have

$$\mathsf{HA}^\omega \vdash \forall f^1 \left(\mathsf{WSup}(\Phi(f)) \to \Pi^0_1\text{-}\mathsf{LEM}(f)\right) \ .$$

As mentioned, [47] proves that WSup implies PBV. This proof has not yet been formalised and refined and the actual status of PBV^- is therefore not known.

Chapter 7

The Strength of Π^0_2-LEM and Σ^0_2-LEM

The results of this chapter will deal with the principles Π^0_2-LEM and Σ^0_2-LEM. As remarked on p. 59, classically equivalent formulations of the Cauchy criterion split up in a constructive setting. This is also the case for the limit superior principle, and we shall see that the two most natural of these formulations intuitionistically correspond exactly to Π^0_2-LEM respectively Σ^0_2-LEM.

7.1 Introducing Principles from Analysis

Definition 7.1. We give two constructively different definitions of the limit superior principle, corresponding to the two following formalisations of "x is limit superior of the sequence of real numbers represented by $f^{0\to 1}$". Positively:

$$\forall k \left[\forall m \exists n >_0 m \left(|x -_{\mathbb{R}} f(n)|_{\mathbb{R}} \leq_{\mathbb{R}} 2^{-k}\right) \wedge \exists l \forall i > l \left(f(i) \leq_{\mathbb{R}} x + 2^{-k}\right)\right] \ . \tag{7.1}$$

Negatively:

$$\forall k \left[\forall m \exists n > m \left(|x - f(n)| \leq 2^{-k}\right) \wedge \neg \forall l \exists i > l \left(f(i) > x + 2^{-k}\right)\right] \ . \tag{7.2}$$

Now define $\mathsf{Limsup}_{\mathrm{pos}}(f)$ by

$$\forall n (f(n) \in_{\mathbb{Q}} [0,1]) \to \exists x \text{ s.t. } (7.1) \ ,$$

and $\mathsf{Limsup}_{\mathrm{neg}}(f)$ by

$$\forall n (f(n) \in_{\mathbb{Q}} [0,1]) \to \exists x \text{ s.t. } (7.2) \ .$$

Remark 7.2. Where operations on reals are used on a rational q, it is to be replaced by its trivial $\mathbb{R}$ representation $\lambda n.q$.

7.2 Existence of the Limit Superior and Π^0_2-LEM

In this section we examine the most natural formulation of the existence of a limit superior — the negative one. It formalises "x is the largest limit point" as "x is a limit point and any $y > x$ is not a limit point", as opposed to moving the negation inside the limit point predicate; "there is a bound for how many points are close to a strictly larger limit point candidate".

[43] gives a Brouwerian counterexample to the lim sup principle based on LPO. In Sect 6.2 we saw that also the monotone convergence principle implies LPO, but intuitively this principle is much weaker than the lim sup principle. To get a stronger reverse direction we therefore make another construction, which in combination with Theorem 4.8 shows this intuition to be true.

Theorem 7.3.

$$\mathsf{HA}^\omega + \mathsf{Limsup}^-_{\mathrm{neg}} \vdash \Pi^0_2\text{-}\mathsf{LEM} \ .$$

Proof. Consider a Π^0_2-LEM instance given by a term $t^{0\to0\to0}$. It is easy to check that the number parameters can be carried around in the proof (as was explicitly done in the proof of Proposition 6.6) causing no problems, and we shall leave them out for the sake of simplicity. Define a sequence of rationals in $[0,1]$ by

$$f(0) :=_0 0 \quad \text{and}$$

$$f(n) :=_0 \begin{cases} 1 -_{\mathbb{Q}} 2^{-a(n)} & \text{if } a(n) > a(n-1) \\ 0 & \text{otherwise} \end{cases} \qquad \text{for } n > 0 \ ,$$

where

$$a(n) := \max\{a \leq n \,|\, \forall a' < a \exists b \leq n (t(a', b) = 0)\} \ .$$

Note that $a(\cdot)$ is non-decreasing.

Let x^1 be the lim sup of f as given by $\mathsf{Limsup}^-_{\mathrm{neg}}$. We show that $\neg(x >_{\mathbb{R}} 0 \wedge x <_{\mathbb{R}} 1)$: Assume $x > 0 \wedge x < 1$, which implies

$$\exists l_0 (x_{l_0+1} +_{\mathbb{Q}} 2^{-l_0} \leq_{\mathbb{Q}} 1_{\mathbb{Q}} \wedge x_{l_0+1} \geq_{\mathbb{Q}} 2^{-l_0}) \ ,$$

so

$$\exists l_0 (x +_{\mathbb{R}} 2^{-l_0-1} <_{\mathbb{R}} 1_{\mathbb{R}} \wedge x >_{\mathbb{R}} 2^{-l_0-1}) \ ,$$

By definition of x we have:

$$\forall k, m \exists n > m \left(|x -_{\mathbb{R}} f(n)|_{\mathbb{R}} \leq_{\mathbb{R}} 2^{-k} \right) \ ,$$

and therefore, with $k = l_0 + 2$,

$$\forall m \exists n > m \big(f(n) \geq x - 2^{-l_0-2} > 2^{l_0-1} - 2^{-l_0-2} = 2^{-l_0-2} \wedge \\ f(n) < 1 - 2^{-l_0-2} \big) \ .$$

This leads to

$$\forall m \exists n > m(a(n) > a(n-1) \wedge a(n) < l_0 + 2) \ ,$$

which of course cannot be the case. Hence $\neg(x >_{\mathbb{R}} 0 \wedge x <_{\mathbb{R}} 1)$. By comparing x_2 to $\frac{1}{2}$ we therefore get $x \leq_{\mathbb{R}} 0 \vee x \geq_{\mathbb{R}} 1$. We can also easily prove $x \leq_{\mathbb{R}} 1 \wedge x \geq_{\mathbb{R}} 0$, and therefore we can prove $x =_{\mathbb{R}} 0_{\mathbb{R}} \vee x =_{\mathbb{R}} 1_{\mathbb{R}}$.

For the two cases $x =_{\mathbb{R}} 1$ and $x =_{\mathbb{R}} 0$, we need the following equivalence:

$$\forall m \exists n > m(a(n) > a(n-1)) \leftrightarrow \forall a \exists b(t(a,b) = 0) \ .$$

For the direction from left to right note that if there are infinitely many points where $a(\cdot)$ is strictly increasing, then $a(n)$ takes arbitrarily large values (proved by induction); in particular, for a given a, there exists n such that $a(n) > a$. The other direction is also elementary.

Assume $x =_{\mathbb{R}} 1_{\mathbb{R}}$. Then, using that the operations provably respect $=_{\mathbb{R}}$,

$$\forall k, m \exists n > m(|1_{\mathbb{R}} -_{\mathbb{R}} f(n)|_{\mathbb{R}} \leq_{\mathbb{R}} 2^{-k}) \ .$$

And therefore $a(n) > a(n-1)$ has to be the case infinitely often. Hence $\forall a \exists b(t(a,b) = 0)$.

Assume $x =_{\mathbb{R}} 0_{\mathbb{R}}$. By definition of x with $k = 2$ we get

$$\neg \forall l \exists i > l(f(i) >_{\mathbb{R}} x + 2^{-2}) \ ,$$

and hence

$$\neg \forall l \exists i > l(f(i) >_{\mathbb{R}} 2^{-2}) \ .$$

We want this to imply $\neg \forall a \exists b(t(a,b) = 0)$, and therefore show the other direction without the negations. That is, assume $\forall a \exists b(t(a,b) = 0)$. By the equivalence above, we get that

$$\forall m \exists n > m(f(n) = 1 - 2^{-a(n)} \wedge a(n) > a(n-1)) \ ,$$

and so

$$\forall m \exists n > m(f(n) \geq 1 - 2^{-1} > 2^{-2}) \ .$$

$\square$

Remark 7.4 *(Strengthening).* There is a term Φ^2 of HA^ω using at most R_0 such that

$$\mathsf{HA}^\omega \vdash \forall f^1(\mathsf{Limsup}_{\mathrm{neg}}(\Phi(f)) \rightarrow \Pi^0_2\text{-}\mathsf{LEM}(f)) \ .$$

Proposition 7.5.

$$\mathsf{HA}^\omega + \mathsf{AC}^{0,0} + \Pi^0_2\text{-}\mathsf{LEM} \vdash \mathsf{Limsup}^-_{\mathrm{neg}} \ .$$

Proof. From $\mathsf{AC}^{0,0}$ and Π^0_2-LEM we get Π^0_2-CA, the comprehension axiom for Π^0_2 formulas. Let the lim sup instance be given by a term t. Π^0_2-CA gives us the characteristic function for

$$\forall k \exists l > k \left(t(l) \in_{\mathbb{Q}} [\tfrac{i}{2^j}, \tfrac{i+1}{2^j}] \right) \quad ,$$

with number parameters j and i. Primitive recursively in this we define the sequence $\{x'_n\}$ such that

$$x'_0 := 0 \quad ,$$

and

$$x'_n := \begin{cases} x'_{n-1} + 2^{-n} & \text{if } \forall k \exists l > k \left(x'_{n-1} + 2^{-n} \leq t(l) \leq x'_{n-1} + 2^{-n+1} \right) \\ x'_{n-1} & \text{otherwise} \quad . \end{cases}$$

$x_n := x'_{n+1}$ defines a Cauchy sequence with rate of convergence 2^{-n}, and therefore a real number x.

We must prove two things about x — firstly that

$$\forall k, m \exists n > m \left(|x - t(n)| \leq 2^{-k} \right) \quad , \tag{7.3}$$

and secondly for a given k

$$\neg \forall l \exists i > l \left(t(i) > x + 2^{-k} \right) \quad . \tag{7.4}$$

For (7.3) we need the following

$$\forall n, k \exists l > k (t(l) \in [x'_n, x'_n + 2^{-n}]) \quad , \tag{7.5}$$

which we show by induction in n: The base case, $n = 0$, is trivial. Now assume n to be such that

$$\forall k \exists l > k (t(l) \in [x'_n, x'_n + 2^{-n}]) \quad .$$

We must prove

$$\forall k \exists l > k (t(l) \in [x'_{n+1}, x'_{n+1} + 2^{-n-1}]) \quad .$$

If $x'_{n+1} = x'_n + 2^{-n-1}$ then we have it by definition. Now assume $x'_{n+1} = x'_n$. Then

$$\neg \forall k \exists l > k (t(l) \in [x'_n + 2^{-n-1}, x'_n + 2^{-n}]) \quad , \tag{7.6}$$

and we aim at

$$\forall k \exists l > k (t(l) \in [x'_n, x'_n + 2^{-n-1}]) \quad .$$

Since Σ^0_1-DNE follows from Π^0_2-LEM[1] (Theorem 4.8) it is enough to prove

$$\forall k \neg\neg\exists l > k(t(l) \in [x'_n, x'_n + 2^{-n-1}]) \ ,$$

and this can be done by contradiction, since $\neg\neg\forall \to \forall\neg\neg$ and three $\neg$'s contract into one. Therefore, assume

$$\neg\forall k \neg\neg\exists l > k(t(l) \in [x'_n, x'_n + 2^{-n-1}]) \ ,$$

which is equivalent to

$$\neg\neg\exists k \neg\exists l > k(t(l) \in [x'_n, x'_n + 2^{-n-1}]) \ ,$$

that in turn is equivalent to

$$\neg\neg\exists k \forall l > k \neg(t(l) \in [x'_n, x'_n + 2^{-n-1}]) \ .$$

By (7.6) and Σ^0_1-DNE again, we also get

$$\neg\neg\exists k \forall l > k \neg(t(l) \in [x'_n + 2^{-n-1}, x'_n + 2^{-n}]) \ .$$

These two give

$$\neg\neg\exists k \forall l > k \neg(t(l) \in [x'_n, x'_n + 2^{-n}]) \ ,$$

by pulling out the double negation and taking the largest k. We reverse the arguments (using that double negation introduction, the reverse of Σ^0_1-DNE, is provable in HA) and get

$$\neg\forall k \exists l > k(t(l) \in [x'_n, x'_n + 2^{-n}]) \ ,$$

which contradicts the induction hypothesis and thereby closes the induction.

We return to the proof of (7.3). By (7.5) and $|x'_n - x| < 2^{-n+1}$ we get

$$\forall n, k \exists l > k(|x - t(l)| < 2^{-n+2}) \ ,$$

hence (7.3).

We have now proved x to be a limit point. What remains is to prove (7.4) — that there can be no larger limit points. Assume that

$$\forall l \exists i > l \left(t(i) > x + 2^{-k}\right) \ .$$

[1]These Π^0_2-LEM instances do not depend on x, only on t, for they are instances of $\forall j \Pi^0_2$-$\mathsf{LEM}(t'[j])$, where t' can be defined from t by primitive recursion and j is a code of a tuple of a natural (to be instantiated with n) and a rational (instantiated with x_n) number. Thus we do not need function parameters.

So, by $|x_n - x| < 2^{-n}$,

$$\forall n > k \forall l \exists i > l(t(i) > x_n - 2^{-n} + 2^{-k} \geq x_n + 2^{-n}) \ ,$$

which expresses that there are infinitely many $t(i)$'s to the right of the n'th interval. By an induction proof similar to the one above (only slightly simpler) one can prove this to be contradictory, and we get (7.4). $\square$

Remark 7.6 *(Strengthening).* There is a term $\Phi^{1\to(0\to 1)}$ of HA^ω using at most R_0 such that

$$\mathsf{HA}^\omega \vdash \forall f^1(\forall n \Pi^0_2\text{-}\mathsf{LEM}((\Phi(f))(n)) \to \mathsf{Limsup}_{\text{neg}}(f)) \ .$$

7.3 Existence of the Limit Superior and Σ^0_2-LEM

We now turn to an analysis of the positive formulation of the existence of a limit superior. Notice in the proof of Theorem 7.3 that the pattern $\neg\forall\exists$, from the last part of the conjunction in the definition of lim sup, gives us the negative part of Π^0_2-LEM — ie. also a $\neg\forall\exists$ pattern. The idea is now to try if our construction also carries the positive pattern $\exists\forall$ into the positive part of Σ^0_2-LEM.

But first we show how $\mathsf{HA} + \Sigma^0_2$-LEM, in a sense, does not distinguish between $\neg\forall\exists$ and $\exists\forall$.

Lemma 7.7.

$$\mathsf{HA} + \Sigma^0_2\text{-}\mathsf{LEM} \vdash \neg\forall a \exists b A_0(a,b) \leftrightarrow \exists a \forall b \neg A_0(a,b) \ ,$$

where A_0 is quantifier-free.

Proof. The implication from right to left is provable in HA.

For the other direction we use the following Σ^0_2-LEM instance:

$$\exists a \forall b \neg A_0(a,b) \vee \neg\exists a \forall b \neg A_0(a,b) \ .$$

The first case is exactly the conclusion. In HA the second case is equivalent to $\forall a \neg\forall b \neg A_0(a,b)$, which in turn implies $\forall a \neg\neg\exists b A_0(a,b)$. Σ^0_1-DNE follows from Σ^0_1-LEM (Theorem 4.8) and we therefore get $\forall a \exists b A_0(a,b)$. By the premise, this implies $\perp$ and therefore also gives us the conclusion. $\square$

Corollary 7.8.

$$\mathsf{HA}^\omega + \mathsf{AC}^{0,0} + \Sigma^0_2\text{-}\mathsf{LEM} \vdash \mathsf{Limsup}^-_{\text{pos}} \ .$$

Proof. Consider a $\mathsf{Limsup}^-_{\mathrm{pos}}$ instance given by a term t^1. We get

$$\mathsf{HA}^\omega + \Sigma^0_2\text{-}\mathsf{LEM} \vdash \Pi^0_2\text{-}\mathsf{LEM}$$

and so

$$\mathsf{HA}^\omega + \mathsf{AC}^{0,0} + \Sigma^0_2\text{-}\mathsf{LEM} \vdash \mathsf{Limsup}^-_{\mathrm{neg}} \ .$$

Using Lemma 7.7 and $|x - \widehat{x}(k)| < 2^{-k}$ we get

$$\begin{aligned}
&\neg\forall l \exists i > l(t(i) > x + 2^{-k}) \to \\
&\neg\forall l \exists i > l(t(i) > \widehat{x}(k) + 2^{-k+1}) \stackrel{\Sigma^0_2\text{-}\mathsf{LEM}}{\to} \\
&\ \exists l \forall i > l(t(i) \leq \widehat{x}(k) + 2^{-k+1}) \to \\
&\ \exists l \forall i > l(t(i) \leq x + 2^{-k+2}) \ ,
\end{aligned}$$

and hence $\mathsf{Limsup}_{\mathrm{pos}}(t)$.

The Σ^0_2-LEM instances used to strengthen $\mathsf{Limsup}_{\mathrm{neg}}(t)$ to $\mathsf{Limsup}_{\mathrm{pos}}(t)$ do not depend on x^1 — the limit superior guaranteed by $\mathsf{Limsup}_{\mathrm{neg}}(t)$ — but only on t in a similar way to the one noted on page 70. Therefore we can get $\mathsf{Limsup}_{\mathrm{pos}}(t)$ without using function variables in the Σ^0_2-LEM instances. ☐

Remark 7.9 *(Strengthening).* There is a term $\Phi^{1\to(0\to1)}$ of HA^ω using at most R_0 such that

$$\mathsf{HA}^\omega \vdash \forall f^1(\forall n \Sigma^0_2\text{-}\mathsf{LEM}((\Phi(f))(n)) \to \mathsf{Limsup}_{\mathrm{pos}}(f)) \ .$$

Proposition 7.10.

$$\mathsf{HA}^\omega + \mathsf{Limsup}^-_{\mathrm{pos}} \vdash \Sigma^0_2\text{-}\mathsf{LEM} \ .$$

Proof. Let Σ^0_2-$\mathsf{LEM}(t)$ be the instance we need to prove. We use the same construction as in the proof of Theorem 7.3 but on $t' := \overline{\mathrm{sg}}(t)$. From the case $\limsup(f) = 1$ we directly get

$$\forall a \exists b(t'(a,b) = 0) \ ,$$

and hence

$$\neg\exists a \forall b(t(a,b) = 0) \ . \tag{7.7}$$

The other case is similar to the corresponding case from Theorem 7.3:

From $\limsup(f) = 0$ with $k = 2$ in the positive lim sup definition we get

$$\exists l_0 \forall i > l_0(f(i) \leq 2^{-2}) \ ,$$

which implies

$$\exists l_0 \forall i > l_0 (a(i) = a(l_0)) \ .$$

Therefore $\forall b \neg (t'(a(l_0), b) = 0)$, and so $\exists a \forall b (t(a, b) = 0)$.

$\square$

Remark 7.11 *(Strengthening).* There is a term Φ^2 of HA^ω using at most R_0 such that

$$\mathsf{HA}^\omega \vdash \forall f^1 (\mathsf{Limsup}_{\mathrm{pos}}(\Phi(f)) \to \Sigma^0_2\text{-}\mathsf{LEM}(f)) \ .$$

Chapter 8

The Strength of Σ^0_1-LLPO and Σ^0_2-LLPO

In this chapter we examine the principles Σ^0_1-LLPO and Σ^0_2-LLPO. As already mentioned in Sect. 5.2.1, the results concerning Σ^0_1-LLPO will shed light on the difference between reasoning constructively about constructive objects and reasoning classically about constructive objects — one mode being represented by (refined) constructive reverse mathematics and the other by (classical) reverse mathematics.

We shall see that our calibration recognises the nondeterminism in BW^- (for a sequence with more than one limit point, BW^- is free to chose between them, so to speak) since BW^- turns out to be equivalent to Σ^0_2-LLPO. This shows that the reasoning that makes weak König's lemma so interesting is also present in BW^- — only on one level higher in the logical hierarchy — *and* a refined constructive reverse mathematics exposes this.

8.1 Introducing Principles from Analysis

This chapter will deal with analytic principles that express properties of uniformly continuous functions. Recall that in the representation we use here, a modulus of uniform continuity ω_f is part of the data of $f \in C[0,1]$ (see p. 25).

Definition 8.1. Let $\mathsf{Max}(f)$ denote the principle stating that if f is defined and uniformly continuous on $[0,1]$ then it attains its maximum:

$$\forall f \in C[0,1] \exists x \in_{\mathbb{R}} [0,1] (f(x) = \sup_{y \in [0,1]} f(y)) \ .$$

A related principle is the intermediate value theorem, stating that if a uniformly continuous function has both negative and positive values, then somewhere it has a root.

Definition 8.2. Define $\mathsf{IVT}(f)$ as

$$\forall f \in C[0,1]\,((f(0) \leq_{\mathbb{R}} 0 \wedge f(1) \geq_{\mathbb{R}} 0) \rightarrow \exists x \in_{\mathbb{R}} [0,1](f(x) =_{\mathbb{R}} 0)) \ .$$

There are two different, common formulations of the Bolzano-Weierstraß principle in the literature. One is that any bounded sequence of rationals has a convergent subsequence. The other states the existence of a limit point. In this treatment we shall look at the simpler, latter one.

Definition 8.3. Define $\mathsf{BW}(f)$ as

$$\forall n(f(n) \in_{\mathbb{Q}} [0,1]) \rightarrow \exists x \in_{\mathbb{R}} [0,1] \forall k, m \exists n > m(|f(n) -_{\mathbb{R}} x|_{\mathbb{R}} \leq_{\mathbb{R}} 2^{-k}) \ .$$

8.2 Attainment of the Maximum

We show the Max^- principle to be equivalent to Σ^0_1-LLPO in the present section.

Proposition 8.4.

$$\mathsf{HA}^\omega + \mathsf{AC}^{0,0} + \Sigma^0_1\text{-}\mathsf{LLPO} \vdash \mathsf{Max}^- \ .$$

Proof. The construction is again performed using bisection. First we prove

$$\forall n \forall j < 2^n \exists i \in \{0,1\} \big(\neg S(j,n) \rightarrow \neg S(2j+i, n+1)\big) \ ,$$

where

$$S(j,n) :\equiv \sup_{y \in [\frac{j}{2^n}, \frac{j+1}{2^n}]} f(y) < \sup_{y \in [0,1]} f(y) \ .$$

Recall that the supremum of f can be defined in HA^ω using at most R_0 due to the fact that f is given with a modulus of uniform continuity (cf. [29]).

For given j, n assume $\neg S(j,n)$. To find an i such that $\neg S(2j+i, n+1)$ we shall use Σ^0_1-LLPO: Assume $S(2j, n+1) \wedge S(2j+1, n+1)$. Then also $S(j,n)$, which is a contradiction and we get $\neg(S(2j, n+1) \wedge S(2j+1, n+1))$. S is a Σ^0_1 formula (it has an $\exists$ quantifier due to the definition of $<_{\mathbb{R}}$) and we therefore, by applying Σ^0_1-LLPO, get $\neg S(2j, n+1) \vee \neg S(2j+1, n+1)$; hence the desired i.

Now $\mathsf{AC}^{0,0}$ gives a binary function finding this i:

$$\exists g^{0\to 1} \leq_{0\to 1} \lambda m, n.1 \forall n, j < 2^n \big(\neg S(j,n) \to \neg S(2j + g(n,j), n+1)\big) \ .$$

Define the function x' to give the left interval endpoints:

$$x'(n) := \begin{cases} 0 & \text{if } n = 0 \\ \frac{2x'(n-1) + g(n-1, x'(n-1)\cdot 2^{n-1})}{2^n} & \text{otherwise} \end{cases} ,$$

and speed it up by $x(n) := x'(n+1)$ so that $x(\cdot)$ defines a Cauchy sequence with rate 2^{-n} and hence a real number. This and the fact that $\forall n \neg S(2^{n+1}x(n), n+1)$ are established by simple induction proofs. It only remains to prove $f(x) \geq \sup_{y\in[0,1]} f(y)$.

Let ω be the modulus of uniform continuity of f. The definition of x ensures that

$$\forall k \left(\sup_{y\in[x(\omega(k)+1), x(\omega(k)+1)+2^{-\omega(k)-2}]} f(y) = \sup_{y\in[0,1]} f(y) \right) ,$$

and therefore, by [56, 6.1.9],

$$\forall k, l \exists y \in [x(\omega(k)+1), x(\omega(k)+1) + 2^{-\omega(k)-2}] (f(y) > \sup_{y\in[0,1]} f(y) - 2^{-l}) \ .$$

Since

$$|y - x(\omega(k)+1)| < 2^{-\omega(k)} \to |f(y) - f(x(\omega(k)+1))| < 2^{-k} \ ,$$

we get

$$\forall k (f(x(\omega(k)+1)) > \sup_{y\in[0,1]} f(y) - 2^{-k+1}) \ .$$

Furthermore

$$|f(x(\omega(k)+1)) - f(x)| < 2^{-k} \ ,$$

since $|x - x(\omega(k)+1)| < 2^{-\omega(k)}$. Combining these two we find that

$$\forall k (f(x) > \sup_{y\in[0,1]} f(y) - 2^{-k+2}) \ ,$$

which by [56, 5.2.11] is equivalent to $\sup_{y\in[0,1]} f(y) \leq f(x)$. $\square$

Remark 8.5 *(Strengthening).* There is a term $\Phi^{1\to(0\to 1)}$ of HA^ω using at most R_0 such that

$$\mathsf{HA}^\omega \vdash \forall f^1 (\forall n \Sigma^0_1\text{-}\mathsf{LLPO}((\Phi(f))(n)) \to \mathsf{Max}(f)) \ .$$

The following proposition is proved in two steps in [56, 6.1.10+5.2.12] by a natural detour over the (non-constructive) principle of totality of $\leq_{\mathbb{R}}$: $\forall a^1, b^1(a \leq_{\mathbb{R}} b \vee b \leq_{\mathbb{R}} a)$.

Proposition 8.6.

$$\mathsf{HA}^\omega + \mathsf{Max}^- \vdash \Sigma^0_1\text{-}\mathsf{LLPO} \ .$$

Examining the proof we again find the following.

Remark 8.7 *(Strengthening).* There is a term Φ^2 of HA^ω using at most R_0 such that

$$\mathsf{HA}^\omega \vdash \forall f^1(\mathsf{Max}(\Phi(f)) \rightarrow \Sigma^0_1\text{-}\mathsf{LLPO}(f)) \ .$$

8.3 The Intermediate Value Theorem

It is well-known that the principle of attainment of the maximum implies the intermediate value theorem. For consider an instance $\mathsf{IVT}(f)$ with $f \in C[0,1]$ and $f(0) \leq 0 \wedge f(1) \geq 0$. Then $g := -|f| \in C[0,1]$, and a maximum for g is easily seen to be a root of f. Hence by Proposition 8.4 we get,

Corollary 8.8.

$$\mathsf{HA}^\omega + \mathsf{AC}^{0,0} + \Sigma^0_1\text{-}\mathsf{LLPO} \vdash \mathsf{IVT}^- \ .$$

Remark 8.9 *(Strengthening).* The mapping $f \mapsto -|f|$ can be implemented by a term of HA^ω. Hence, there is a term $\Phi^{1\rightarrow(0\rightarrow1)}$ of HA^ω using at most R_0 such that

$$\mathsf{HA}^\omega \vdash \forall f^1(\forall n \Sigma^0_1\text{-}\mathsf{LLPO}((\Phi(f))(n)) \rightarrow \mathsf{IVT}(f)) \ .$$

On the other hand results from recursive mathematics might suggest that IVT^- is strictly weaker than Max^-:

It is known that for a computable function f, $f(0) \leq 0$ and $f(1) \geq 0$, there exists (in a classical sense) a computable root. We sketch the proof. There are two cases: Either f has a root in some rational number, or it does not. The rational numbers are computable, which closes the first case. In the second case we split the interval in two and evaluate f at c, the point of division. This is a rational point and so we know that $f(c) \neq 0$ — hence $f(c) < 0 \vee f(c) > 0$. In either of these subcases we end in a situation like the initial. Thus we can define a Cauchy sequence converging towards a root of f. For full details see [46, 0.6.8].

This rather positive result does not have a counterpart for the Max^- principle. In [51] a computable function with no computable point of maximum is constructed. In spite of this mismatch between Max^- and IVT^-, the following proposition shows the two to be equivalent over $\mathsf{HA} + \mathsf{AC}^{0,0}$.

Proposition 8.10.
$$\mathsf{HA}^\omega + \mathsf{IVT}^- \vdash \Sigma^0_1\text{-}\mathsf{LLPO} \ .$$

The proof is a standard example of a Brouwerian counterexample and can be found in [56, 6.1.2] or even in Brouwer's own [5].

Remark 8.11 *(Strengthening).* There is a term Φ^2 of HA^ω using at most R_0 such that
$$\mathsf{HA}^\omega \vdash \forall f^1(\mathsf{IVT}(\Phi(f)) \to \Sigma^0_1\text{-}\mathsf{LLPO}(f)) \ .$$

As an immediate consequence of the proposition we have the first part of the following.

Corollary 8.12. *(i)* $\mathsf{HA}^\omega + \mathsf{AC}^{0,0} \vdash \mathsf{IVT}^- \leftrightarrow \mathsf{Max}^-$

(ii) $\mathsf{HA}^\omega + \mathsf{QF}\text{-}\mathsf{AC}^{0,0} \nvdash \mathsf{IVT}^- \to \mathsf{Max}^-$

Remark 8.13. For two schemata S_1 and S_2, $\mathsf{H} \vdash \mathsf{S}_1 \to \mathsf{S}_2$ means "for all terms $t_2[n]$ of HA there exists a term $t_1[m]$ of HA such that $\mathsf{H} \vdash \forall m \mathsf{S}_1(t_1[m]) \to \forall n \mathsf{S}_2(t_2[n])$".

Proof. (i) is immediate. (ii): If it were the case, then also
$$\mathsf{PA}^\omega + \mathsf{QF}\text{-}\mathsf{AC}^{0,0} \vdash \mathsf{IVT}^- \to \mathsf{Max}^- \ .$$

But $\mathsf{PA}^\omega + \mathsf{QF}\text{-}\mathsf{AC}^{0,0} \vdash \mathsf{IVT}^-$ (cf. [49, II.6.6]) and $\mathsf{PA}^\omega + \mathsf{QF}\text{-}\mathsf{AC}^{0,0} \nvdash \mathsf{Max}^-$, since
$$\mathsf{HEO} \models \mathsf{PA}^\omega + \mathsf{QF}\text{-}\mathsf{AC}^{0,0} \ \text{ and } \ \mathsf{HEO} \not\models \mathsf{Max}^- \ .$$
([54, 2.4.11] and the above mentioned [51].) $\square$ $\square$

One might think that the project of this paper, which could be seen as a first attempt to do a refined, intuitionistic reverse mathematics, only distinguishes more mathematical principles than classical reverse mathematics. The argument being, that intuitionistic logic as a restriction of classical logic proves fewer equivalences, and the refinement only contributes to this. From the example above we see that this is not the case; classical reverse mathematics distinguishes Max^- and IVT^-, but intuitionistically they should be identified.

The corollary pinpoints the reason for this. Identifying Max^- with IVT^- requires application of axiom of choice for arithmetical formulas, $\mathsf{AC}^{0,0}_{\mathrm{ar}}$. As $\mathsf{AC}^{0,0}_{\mathrm{ar}}$ only makes the interpretation of the logical connectives and quantifiers of intuitionistic logic explicit, $\mathsf{HA}^\omega + \mathsf{AC}^{0,0}_{\mathrm{ar}}$ is a reasonable constructive and robust system; we even saw in Chap. 3 that the full AC principle preserves a constructive interpretation. Adding $\mathsf{AC}^{0,0}_{\mathrm{ar}}$ to a classical system (like PA^ω), on the other hand, has the unwanted consequence that both Max^- and IVT^- become provable (and therefore trivially equivalent) along with full arithmetical comprehension — which in turn gives much stronger theorems like Bolzano-Weierstraß and existence of lim sup.

So IVT^- is provable in RCA_0 and therefore "constructive" in the sense of [49]. But we saw that in our intuitionistic setting, it allowed to derive Σ^0_1-LLPO and therefore cannot be constructive. The reason is that the proof sketched above does not give a method that finds the root.

To have classical reverse mathematics recognise the intermediate value theorem as non-constructive, it is necessary to either formulate both principles such that they incorporate full uniformity, which for the intermediate value theorem amounts to

> there is a functional mapping continuous functions f on $[0,1]$ with $f(0) \leq 0 \wedge f(1) \geq 1$ to a root of f.

Or as a compromise,

> for a sequence of continuous functions f_n on $[0,1]$ with $f(0) \leq 0 \wedge f(1) \geq 1$ there exists a sequence x_n of roots.

The first approach leaves the domain of second order arithmetic, and is therefore not treated in [49]. It is covered in [25, 3.14], which proposes the so-called "higher order reverse mathematics".

The compromise is treated in [49, Exercise IV.2.12] where it is found to be equivalent to WKL_0 over RCA_0.

8.4 Bolzano-Weierstraß

Proposition 8.14.

$$\mathsf{HA}^\omega + \mathsf{AC}^{0,0} + \Sigma^0_2\text{-}\mathsf{LLPO} \vdash \mathsf{BW}^- \ .$$

Proof. The proof is similar to that of Proposition 7.5.

We consider a BW^- instance given by a term t^1 satisfying $\forall n(t(n) \in_{\mathbb{Q}} [0,1])$.

As in Proposition 7.5 and Proposition 8.4 we follow the classical bisection approach. Compared to Proposition 7.5 the difference is that we now cannot be sure to always find the rightmost interval with infinitely many points, and therefore we cannot always find the largest limit point — the limit superior.

We make a definition similar to the one used in the proof of Proposition 8.4:

$$S(j,n) := \forall a \exists b > a(t(b) \in_{\mathbb{Q}} [\tfrac{j}{2^n}, \tfrac{j+1}{2^n}]) \ ,$$

and we shall prove

$$\forall n \forall j < 2^n \exists i \in \{0,1\}(S(j,n) \to S(2j+i, n+1)) \ .$$

For given n, j assume $S(n,j)$. We aim at using Σ^0_2-LLPO to find an i such that $S(2j+i, n+1)$. Therefore we assume

$$\exists a \forall b > a(t(b) \notin [\tfrac{2j}{2^{n+1}}, \tfrac{2j+1}{2^{n+1}}]) \wedge \exists a \forall b > a(t(b) \notin [\tfrac{2j+1}{2^{n+1}}, \tfrac{2j+2}{2^{n+1}}]) \ .$$

Then also

$$\exists a' \forall b > a'(t(b) \notin [\tfrac{2j}{2^{n+1}}, \tfrac{2(j+1)}{2^{n+1}}]) \ ,$$

which is a contradiction. Hence

$$\neg \left(\exists a \forall b > a(t(b) \notin [\tfrac{2j}{2^{n+1}}, \tfrac{2j+1}{2^{n+1}}]) \wedge \exists a \forall b > a(t(b) \notin [\tfrac{2j+1}{2^{n+1}}, \tfrac{2j+2}{2^{n+1}}])\right) \ .$$

Σ^0_2-LLPO now gives us an $i \in \{0,1\}$ such that $S(2j+i, n+1)$. By $\mathsf{AC}^{0,0}$ we get a binary function g^1 finding this i, and by primitive recursion we define the functional $x(\cdot)$ as in the proof of Proposition 8.4 to give left interval endpoints with rate 2^{-n}:

$$x'(n) := \begin{cases} 0 & \text{if } n = 0 \\ \frac{2x'(n-1)+g(n-1, x'(n-1)\cdot 2^{n-1})}{2^n} & \text{otherwise} \ , \end{cases}$$

and $x(n) := x'(n+1)$. Let x denote the real number thus defined. By induction we easily get $\forall n S(2^{n+1}x(n), n+1)$, ie.

$$\forall n, a \exists b > a(t(b) \in [x(n), x(n) + 2^{-n-1}]) \ . \tag{8.1}$$

We still need to prove that x is a limit point, ie.:

$$\forall n, a \exists b > a(|x - t(a)| \leq 2^{-n}) \ .$$

This follows, as in the proof of Proposition 7.5, from the fact that $x(\cdot)$ is Cauchy with rate 2^{-n} and (8.1). □

Remark 8.15 *(Strengthening).* There is a term $\Phi^{1\to(0\to1)}$ of HA^ω using at most R_0 such that

$$\mathsf{HA}^\omega \vdash \forall f^1(\forall n \Sigma^0_2\text{-}\mathsf{LLPO}((\Phi(f))(n)) \to \mathsf{BW}(f)) \ .$$

Theorem 8.16.

$$\mathsf{HA}^\omega + \mathsf{BW}^- \vdash \Sigma^0_2\text{-}\mathsf{LLPO} \ .$$

Proof. Let the Σ^0_2-LLPO instance be given by two terms t_1 and t_2, such that the premise in Σ^0_2-LLPO is satisfied, ie.

$$\neg\left(\exists a \forall b (t_1(a,b) = 0) \wedge \exists a \forall b (t_2(a,b) = 0)\right) \ .$$

Define

$$a(n) := \max \left\{ a \leq n \;\middle|\; \begin{array}{r} a = 0 \vee \forall a' \leq a \exists b \leq n (t_1(a',b) \neq 0) \vee \\ \forall a' \leq a \exists b \leq n (t_2(a',b) \neq 0) \end{array} \right\}$$

and then

$$f(n) := \begin{cases} 2 + 2^{-a(n)} & \text{if } \forall a' \leq a(n) \exists b \leq n (t_1(a',b) \neq 0) \\ 2^{-a(n)} & \text{otherwise} \ . \end{cases}$$

Let x be the limit point guaranteed by $\mathsf{BW}^-(f)$. We are going to prove $x \geq 0 \wedge x \leq 2 \wedge \neg(x > 0 \wedge x < 2)$. As in the proof of Theorem 7.3 we then get $x = 1 \vee x = 2$. Finally we shall show

$$[x = 2 \to \forall a \exists b (t_1(a,b) \neq 0)] \wedge [x = 0 \to \forall a \exists b (t_2(a,b) \neq 0)] \ .$$

But first assume $x > 2$. We then easily get a k_0 such that

$$\forall m \exists n > m (f(n) \geq 2 + 2^{-k_0}) \ .$$

Hence

$$\forall m \exists n > m (a(n) \leq k_0) \ ,$$

which by the monotonicity of $a(\cdot)$ gives

$$\forall n (a(n) \leq k_0) \ ;$$

that is,

$$\forall n (\exists a \leq k_0 + 1 \forall b \leq n (t_1(a,b) = 0) \wedge \exists a \leq k_0 + 1 \forall b \leq n (t_2(a,b) = 0)) \ . \quad (8.2)$$

Now, in PA we have

$$\forall n \exists a \leq k_0 + 1 \forall b \leq n (t_1(a,b) = 0) \to [\forall a \leq k_0 + 1 \exists b (t_1(a,b) \neq 0) \to \perp] \ ,$$

by CP, the collection principle (cf. Definition 2.16). So by the negative translation we have

$$\forall n \exists a \leq k_0 + 1 \forall b \leq n (t_1(a,b) = 0) \to [\neg \forall a \leq k_0 + 1 \neg\neg \exists b (t_1(a,b) \neq 0)] \ ,$$

in HA. Using that $\neg\forall\neg\neg\exists$ gives $\neg\neg\exists\forall\neg$ in HA, we get

$$\neg\neg\exists a \leq k_0 + 1 \forall b (t_1(a,b) = 0) \wedge \neg\neg\exists a \leq k_0 + 1 \forall b (t_2(a,b) = 0) \ ,$$

from (8.2). By pulling out the double negations we get a contradiction to the premise of Σ^0_2-LLPO. The same way we can get a contradiction from $x > 0 \wedge x < 2$.

Since $x \geq 0$ is trivial, we now, by the preliminary arguments, have $x = 0 \vee x = 2$.

Assume $x = 2$. We get

$$\forall k, m \exists n > m (|2 - f(n)| \leq 2^{-k}) \ ,$$

hence,

$$\forall k \geq 2 \forall m \exists n > m (2^{-a(n)} \leq 2^{-k} \wedge \forall a' \leq a(n) \exists b \leq n (t_1(a', b) \neq 0)) \ ,$$

which implies

$$\forall a \exists b (t_1(a,b) \neq 0) \ .$$

If the "otherwise" case in the definition of f occurs, we must have $\forall a' \leq a(n) \exists b \leq n (t_2(a', b) \neq 0)$ by definition of $a(n)$. Analogously we thus get $\forall a \exists b (t_2(a,b) \neq 0)$ if $x = 0$. $\square$

Remark 8.17 *(Strengthening).* There is a term Φ^2 of HA^ω using at most R_0 such that

$$\mathsf{HA}^\omega \vdash \forall f^1 (\mathsf{BW}(\Phi(f)) \to \Sigma^0_2\text{-}\mathsf{LLPO}(f)) \ .$$

Chapter 9

Concluding Remarks

We have provided a series of equivalences between semi-classical logical principles and formalised theorems of analysis, and discussed the proof theoretic interpretation of the logical principles. For each of the principles in the first two levels of the hierarchy, except the DNE-principles, an equivalent analytic principle has been found. On the other hand, we have in all cases except one only provided one equivalence in an attempt to do a breadth-first investigation of refined constructive calibration theory.

In this final chapter we look at the results in the light of our motivations and discuss the next level of the investigation.

Motivations Revisited

In Chap. 1 we categorised our motivations in four groups. Here we go through the groups and conclude on how our results shed light on the motivations and vice versa.

Limit Computable Mathematics. A weak part of the hierarchy that was established in [1] has been found to reflect a substantial amount of mathematics. Also, the generalised constructivity provided by LCM provides a way of understanding certain restricted forms of classical reasoning. First of all, we have seen that $\mathsf{PCM}^-_{\mathrm{ar}}$ is limit realizable in that it is implied by Σ^0_1-LEM over HA. But we have also seen that $\mathsf{PCM}^-_{\mathrm{ar}}$ entails Σ^0_1-LEM and therefore requires a very strong form of limit computability.

BW^- was found to be equivalent to Σ^0_2-LLPO and therefore generally not limit realizable. Still, the technique used to tell the levels in the logical hierarchy apart provides an understanding of BW^-. We saw that $\mathsf{HA} + \Sigma^0_2$-LLPO reduces to $\mathsf{HA}[g] + (\varepsilon) + \Sigma^0_1$-$\mathsf{LLPO}[g]$, where (ε) expresses that g is a Σ^0_1-LEM

oracle — hence limit computable. That is, BW^- features as weak König's lemma only on one level higher in the hierarchy.

Reverse Mathematics and Proof Mining. The logical hierarchy provides a robust system for calibrations. We saw that some principles that are identified in reverse mathematics, are distinguished in our context. We also argued that the new distinctions were valuable in that it is motivated by LCM and proof mining. On the other hand, new identifications, IVT^- and Max^-, were made, and we showed that the classical distinction was due to the lack of weak choice axioms in RCA_0.

The informal constructive reverse mathematics of [21] identifies for example PCM and BW because principles are considered with function parameters. We have found that using the restriction to function parameter-free instances to prohibit iteration of the principles, reveals that BW^- is very much different from PCM^- — as also the refinement in the classical setting shows. The difference between PCM^- and BW^- is akin to that between intuitionistically provable theorems and weak König's lemma. And it has turned out that on the level of logic, the generalised weak König's lemma can be expressed in the Σ^0_n-LLPO principles.

The discussion in Sect. 4.3 on the various proof interpretations and their connection to the logical hierarchy translates to the mathematical by the equivalences. So a proof system including the intermediate value theorem *and* Markov's principle has the bound extraction property as does also a system with the weak supremum principle. Note that the oracle interpretation on for instance level 2 corresponds to computability in the jump $\emptyset'$ (cf. [48]), so for instance from a proof using BW^- *and* Σ^0_2-DNE one can extract bounds that are computable in $\emptyset'$.

Constructivism and Mathematics. As it is noted in reverse mathematics, large parts of mathematics can be carried out in very weak proof systems. This thesis suggests that this is true also when weak is taken in the sense of the semi-classical principles since the diversity of theorems studied here only require principles of level ≤ 2.

The three principle-types Σ^0_n-LLPO, Π^0_n-LEM, Σ^0_n-LEM seem to capture the non-constructivity in mathematics. It is note-worthy that the Σ^0_n-DNE principles are not represented in any equivalence. It did feature in the discussion of $\mathsf{PCM}^-_{\mathrm{ar}}$ (Sect. 6.3), where we saw it to be a sufficient but not necessary principle.

Future Directions

There are still two very concrete problems that need to be addressed. One is the status of PBV^- in our context, which is yet to be sorted out. The other has to do with IVT.

We know that for a computable continuous function $f : [0,1] \to \mathbb{R}$ with $f(0) \leq 0 \land f(1) \geq 0$ there exists a computable root, but we can in general not find the root given f; this is reflected by the fact that the proof of the existence sketched in Sect. 8.3 uses Σ^0_1-LEM (and later Σ^0_1-DNE). On the other hand we know that using only Σ^0_1-LLPO, we can find a root without requiring it to be computable. Yet, it is not known if Σ^0_1-LEM is actually needed above — that is, does the intermediate value theorem with the requirement that the root is computable imply Σ^0_1-LEM in our setting? If so, this would mean that if one insists on getting a computable root then a $\emptyset'$ oracle is needed, but just to find *some* root, a so-called low-degree[1] oracle is sufficient.

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[1] A degree $\mathbf{a}$ is low if $\mathbf{a}' = \emptyset'$.

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