Errata and Addendum to "Mathematics based on Learning", Ver.2002.11.29

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1 Page 7

1. Line 13 from the bottom: Shaprio's → Shapiro's
2. Line 6 from the bottom: enumerable ideals → all ideals

2 Page 15

1. Line 9 from the top: Σ^0_2-LEM → Σ^0_1-LEM
2. Line 13 from the top: $y = |x|$ over real numbers is equivalent to Σ^0_2-LEM. → $y = [x]$ over real numbers is equivalent to Σ^0_1-LEM, but another formulation is equivalent to Π^0_1-LEM.

3 Section 4.4

Section 4.4 is now obsolete by some very interesting new findings since the paper sent to the editor. The problem of learning theoretic interpretation of WKL is nearly solved in a surprising way. The solution had been prepared by V. Lifschitz' and van Oosten's works in metamathematics of constructive mathematics!

After the paper was dispatched to the editor, the correct formulation of WKL in LCM-setting turned out to be just the ordinary WKL. It is not necessary to restrict the binary trees to be computable. Instead, the functions should be interpreted as defined by the sets in a model of WKL in which all sets are uniformly coded by a fixed low set. Such a model construction using the low basis theorem is found in Simpson's work [1], Theorem VIII.2.17. A Kleene style realizability based on Simpson's model is easily defined and is sound for an intuitionistic system with WKL. Π^0_1-LEM does not hold in the interpretation, as it is based on low degrees which is weaker than Π^0_1.

Even more intrinsic semantics of WKL is Lifschitz-van Oosten realizability interpretation. V. Lifschitz [2] introduced a realizability interpretation for a metatheoretical investigation of constructive mathematics. Later, van Oosten [3] extended it to higher order systems. An introduction can be found in Troelstra's article [4].

It turned out that van Oosten's version of the interpretation of formal systems with function variables is a good semantics of WKL. Even some LCM-like semi-classical principles were considered by van Oosten. Lifschitz and van Oosten do not seem to be aware of linkage of their work to learning theory. Nonetheless, their works give a very simple and interesting model of "computation with refutation." In Lifschitz' interpretation, finite sets of pairs of possible values and Π^0_1-conditions is considered. A value is a real value of the set if and only if the paired condition is correct. Since the condition is Π^0_1, all "incorrect possible value" are eventually refuted. In this way, Lifschitz' possible value sets are correctable in the limit and could be considered to be a very simple kind of inductive inference. We may call this paradigm as "computation with refutation" or "Popperian computation".

It is very likely that we can define a non-deterministic computation system (non-deterministic programming language) based on Lifschitz' idea, and the realizability based on the computation system would serve as a fine system for proof animation of mathematical proofs using WKL. As reverse mathematics project has shown [1], WKL is surprisingly fertile from mathematical point of view. Although reverse mathematics assumes classical reasonings, they are often inessential. For example, Simpson's derivation of the completeness theorem of first order predicate logic is constructive relative to WKL. Thus, a semantics based on Lifshitz computation system would provide a proof animation of the completeness theorem of first order predicate logic.

In Lifschitz-van Oosten model of WKL, Π^0_1-LEM does not hold. Kohlenbach [5] has given another interpretation in which WKL holds but Π^0_1-LEM does not hold. The relation of Kohlenbach's intepretation to Lifschitz-van Oosten interpretation should be investigated, since his interpretaions have important applications to numerical analysis and the model of Popperian computation is expected to be related to limit computation in practical analysis.

Another interesting problem is relationship between the realizability based on Simpson's model and Lifschiz-van Oosten style realizability. It is very likely that the relationship is the one of generic construction and forcing.

References

1. Simpson, S. G.: Subsystems of Second Order Arithmetic, Springer, 1999
2. Lifschitz, V.: CT_0 is stronger than CT_0!, Proceedings AMS, 73, 101–106.
3. van Oosten, J.: Lifschitz' Realizability, Journal of Symbolic Logic, Volume 55, Issue2, 805–821.
4. Troelstra, A.: Realizability, in Handbook of Proof Theory, S. Buss (ed.), Elsevier, Amsterdam, 1998, 407–474
5. Kohlenbach, U.: Proof Interpretations and the Computational Contents of Proofs (draft in progress), BRIC, University of Aarhus, available at `http://www.brics.dk/~kohlenb/`