Hilbert and Computation

Influences of Hilbert's early algebraic works to his studies in the foundations of mathematics

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A forthcoming paper

- We are now working on a paper, which shows how Hilbert's foundational works were influenced in his early algebraic works, especially by the problem of computation appeared in it.

- We will show that it seems that his foundational studies were ***modeled*** on his works in invariant theory in 1880-90's.
- We can draw some interesting conclusions from this interpretation.
- In this talk, we report some specific conclusions among them.

Completeness and Decision methods of mathematics

- Turing's observation
  - If a formal system is formally complete, then we have a decision method for the system via brute force search.
  - Thus, if a theory is undecidable, then there is no complete formal system.

Ha ments in "R Kurt Goedel"

Turing's observation

If Hil aware of this connection at that time, he would probably have been more skeptical about the existence of complete formal systems in the two cases, since presumably he would have n doubt about the decidable…

Theories of natural numbers and real numbers

Conclusion 1

- Even existence of decision method of mathematics in Turing's sense would not affect Hilbert's conviction of completeness much.
- Reason: He knew existence of decision method does not trivialize mathematics from his experience in invariant theory.

Conclusion 2

- However, it is more likely that decision methods of mathematics in Hilbert's sense would be rather restrictive than the decision methods in Turing's sense.
- Brute force search would not be "computation" in his sense.
- Thus, Turing's observation did not apply to his case.

Conclusion 3

- Problem of computability or constructivity was Hilbert's great concern.
- The concern was initiated by his study of invariant theory.

Conclusion 3 (continued)

- It's is likely that he modeled the problems of completeness and decision methods of mathematics on his experience on the solution of Gordan problem in 1880's-1890's.

Conclusion 4

- On this interpretation, his and Bernays' some seemingly wired statements, which puzzled Hao Wang, are natural.

Hilbert's first encounter of the problem of finiteness in mathematics.

- When he was an unknown young man, he was studying invariant theory.
- The central problem of invariant theory at the time was Gordan's problem.
- It was 1887… * *

1886-87 Lecture notes on invariant theory (1)

- Gordan's problem was to find a kind of basis for each effectively given set of algebraic formulas. The basis must be a finite set.
- It was solved in special cases, but not for the general case. Mathematicians were mainly attacking the problem by trying an algorithm giving the solution.

1886-87 Lecture notes on invariant theory (2)

- It resembles the problem to find a finite axiom system for a given theory.
- In the winter semester 1886-87, still in Koenigsberg, Hilbert gave lectures on the subject.
- He wrote like this…..

1886-87 Lecture notes on invariant theory (3)

- Gordan's method is too complicated for actual computation and can be carried out only for small cases.
- The important and fundamental problem is only the finiteness of the system.
- Cod Ms. Hilbert 521 p.193, p.194

Hilbert's solution

- He gave a general solution to the problem in 1888.
- The solution was based on a method which is not recursive but limiting recursive in the sense of the algorithmic learning theory, thus was no computable.
- Gordan criticized it as "not mathematics, but theology"

Hilbert's defense

- Hilbert defended his method in a rather emotional way in a letter to F. Klein, who is the editor-in-chief of Math. Ann. in which the paper should appear and Gordan worked as a reviewer of the paper.

Computational solution

- Nonetheless, a few year later, he gave a computational solution to the same problem based on an entirely different method known as Nullstellenzsatz.
- The algorithm is now known as Hilbert's algorithm in the circle of computer algebra.

Hilbert's offence

- Even after establishing his fame, he repeatedly mention on the affair with Gordan.
- Axiomatisches Denken, 1920 papers on proof theory, etc.
- They are all published about 40 years later after the affair.
- The affair must be very impressive for him.

Hilbert's 1893 lectures on invariant theory

in 2002verison "1893" was mistakenly wrote as 1887

- Among literatures in which he mentioned the affair, 1893 lectures on invariant theory in Goettingen is especially interesting. It was just a few months prior to Cantor's letter to him telling the first set theoretical paradox.
- 3 copies kept in Cornell and Goettingen Math. Institut and English translation has appeared

Three levels of existence theorem

- He explains there are 3 levels with existence theorems

1. Proving the existence: his first solution
2. Determining how many operations are needed at the most to carry out the assertion of the theorem: his second solution
3. Actually carry out the computation

Hilbert's illustration of the three steps

- Assume there is ten consecutive ones 1111111111 in the decimal expansion of π.

1. First, prove the existence.
2. Second, find a number *N* of which one knows that there are 1111111111 before the *N*th decimal of π.
3. Third, calculate the actual occurrence.

Other evidences of algebraic infulences

- Similarilty of his axiomatics foundations and Kronecker's foundations of pure mathematics by his ModulTheoire. Ideal elements = Kronecker's indeterminates
- Try and error "learning" process of ε-substituion methods and his first solution of Gordan problem.
- Etc.etc……

The 2nd level and Turing's observation

- From Turing's observation, the second and third levels are achieved just by brute force search.
- However, this is not what mathematicians, especially applied mathematician thinks.
- The second level has a great practical and mathematical importance.

Computation in Hilbert's sense (1)

- Modern computer science now knows Turing's computation is not computation in real life. NP-problems etc. are computable but not non-computable in real life.
- Hilbert knew such a phenomenon as he was a great calculator in his youth.
- Hilbert's formula table for his dissertation *

Computation in Hilbert's sense (2)

- Thus, for him, a decision method of mathematics in Turing's sense, or even restricted sense, e.g. primitive recursive one, would be only "*method in principle*."

Computation in Hilbert's sense (3)

- A phrase in Hilbert and Ackerkmann book suggest that they believe a decision method for predicate calculus
- But, they wrote that practical application of such method might be "illusolich" after its complicated computation steps.

Computation in Hilbert's sense (4)

- We now know even Fermat's theorem is automatically proved by machines *in principle*.
- But, we also know that it does not trivialize mathematics.
- Hilbert would know this from his experience

Other evidences of algebraic influences

- Similarility of his axiomatics foundations and Kronecker's foundations of pure mathematics by his ModulTheoire.
- try and error "learning" process of ε-substitution methods and his first solution of Gordan problem.

Computation, Algebra, Hilbert

- It's very likely Hilbert modeled logic on algebra as he did in 1900's.
- And, the algebra he was involved is deeply related to the study of contemporary computer algebra.
- He believed mathematics very deeply. We should examine his thoughts from his mathematics, especially, algebra.

Computatio… Conceptu…

- D. Laugwt… mathemati… from comp… initiated by…
- Hilbert was "Le… …n the revolution.
- Hilbert's foundational works should be re-examined from this point of view.

Laugwitz ignored "sociological" aspects of the revolution, and so underestimate set theory and related foundational studies which had "sociologically" and "technically" played the central role in it.