Hilbert and Computation
Influences of Hilbert's early algebraic works to his studies in the foundations of mathematics

A first draft of the first section of a paper

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Abstract

Hilbert's finitism in his program of the 1920's was a countermeasure against the to his axiomatic foundations of mathematics in 1900's by Poincare and Brouwer. However, it is not correct to regard it just as a countermeasure to these criticisms. His respects to finitistic and computational tendencies in mathematics had already existed even in 1890's. He had to ponder on the problem of computational vs non-computational methods in mathematics that he faced Gordan's criticisms against the work on invariant theory and more importantly Kronecker's finitistic philosophy. Although he championed non-constructive/non-computational methods as the mathematics of the new age, he also gave deep respects to and had sympathy with computational methods not from ontological/philosophical but from mathematical point of view. It is quite plausible that his foundational theories in 1900's and 1920's are modeled on these early algebraic works and Kronecker's foundations of mathematics with Modulsysteme.

1 Hilbert and foundational studies

Recently, Hilbert's unpublished lecture notes have been under detailed historical investigations, and some new insights have been reported.

A thorough illustration of Hilbert's program based on such new investigations can be found, e.g., in Mancosu[5]. In this paper, we present one of such investigations. Our aim is to show how storngly his early algebraic works influenced his works in the foundations of mathematics, that is, how important "the problem of computation in mathematics" was for Hilbert.

Hilbert stressed the finitary character of axiomatic systems in his famous Paris talk. Although finitistic-formalistic foundations of mathematics appeared only in his later program in 1920's, it is apparent that the finiteness of axiomatic systems played a central role even in his early axiomatics at the turn of the 20th century. The later finitism can be explained as a countermeasure to Poincare's criticism of the vicious circle in his idea of consistency proofs and Brouwer's intuitionistic criticism of classical principles. However, the early finitistic tendencies in Paris are before these criticisms. *When and how did his finitistic tendency come in?*

If axiomatic or abstract mathematics is the mathematics of the 20th century, Hilbert would be the first 20th century mathematician. However, he was also one of the last 19th century mathematicians. Although he strongly opposed to Kronecker and championed Cantor, he also respected Kronecker. His respect to the great mathematician of his younger time can be heard even 39 years after Kronecker's death in his famous talk at Königsberg September 1930.

Although he was a wizard of non-constructive set theoretical methods in mathematics, he also knew the importance of constructive or computational aspects of mathematics. Three years after the publication of his astonishing non-constructive proof of Gordan problem in 1890, he even published a computational version of the same theorem. Then he ceased his investigation of invariant theory saying all works had been done. Throughout his life, Hilbert used these two solutions of the same problem to illustrate difference of non-constructive and constructive proofs, and stressed importance of both of them. A notable instance is his 1897 lectures on invariant theory [4]. In the lecture dating July 12th, Hilbert talked on three levels of proofs of mathematical existence theorems. First, pure existence proof. Second, estimation of numbers of operations to find the solution. Third, actual calculation of the solution.

He pointed out that the second level was the thing Kronecker had particularly emphasized, and he said what he had done in invariant theory were the first and the second levels of these three. He presented an interesting example to illustrate difference of these two levels, which reminds us Brouwer's later discussion on occurrence of 123456789 in the decimal expansion of π. It would show that Hilbert had already deep insight on computational aspects of mathematics much earlier than Brouwer. Hilbert related it to Kronecker.

It seems that he thought the first two levels are equally important. However, Hilbert was then ontologically non-finitistic. He did not find any problem in classical non-computational mathematics, that is, the first level proof, although he found significance of Kroneckerian computational aspect of mathematics, that is, the second level proof, from mathematical point of view just as his second paper. The theme was repeated in "Axiomatisches Denken" (1917) and in some proof theoretic papers in 1920's.

However, a few months later after the lectures, Cantor told him the first set theoretical antinomy. Any record of his reaction by himself is not known. However, Bernays reports that Hilbert even thought Kronecker might have been right after antinomy of set theory. He had an extensive discussion on existence and consistency with Cantor in the next year 1898. In the winter semester of 1898/1899, he wrote the formalistic thesis "consistency=existence" in his lecture notes on the foundations of geometry. Then, he did not mention finiteness of axiom systems. In the fall, he started talking about *finiteness* of axiom systems and the derivations from them in the paper "Über den Zahlbegriff" 1900. In the Paris talk next year, his stress on finiteness became even clearer.

Then, Zermelo and Russell found an even sharper version of antinomy of set theory, which is said to have made Hilbert think even consistency of integer arithmetic (Bernays[1]). Hilbert presented his pre-proof theoretic paper at the third International Congress of Mathematicians at Heidelberg, 1904, and his rather serious concerns about the foundations of mathematics and logic in the first half of the 1900's have been explored by recent investigations,e.g., Zach[9].

This series of historic events starting from his lecture on invariant theory might show the evolution of Hilbert's thoughts on foundations of mathematics. In the course, he always referred to Kronecker. Kronecker's algebraic works were technically important means in Hilbert's invariant theory, but Hilbert made a great conceptual leap, which Kronecker had never thought about.

Kronecker restricted every thing finite. Any "general notion", e.g. number series in general, was infinite and so non-mathematical for him. Thus, Dedekind's ideal was meaningless to him, unless a particular ideal be shown to have a finite presentation, i.e. basis. Hilbert showed that, in modern terminology, any general countable ideal of polynomials with n-variables has finite basis. He called it *General Finiteness Theorem* now known as Hilbert's finite basis theorem. Of course, he had to use a non-constructive method, which was once accused as "not mathematics but theology" by P. Gordan.

Reid[6] writes that Bernays explained as follows:

> "For Hilbert's program," he explains, "experiences of his scientific career (in fact, even out of his student days) had considerable significance; namely, his resistance to Kronecker's tendency to restrict mathematical methods and, particularly, set theory. Under the influence of the discovery of the antinomy in set theory, Hilbert temporarily thought that Kronecker had probably been right there. But soon he changed his mind. Now it became his goal, one might say, to do battle with Kronecker with his own weapons of finiteness by means of a modified conception of mathematics... (p.173 [6]. See also the footnote e, p.210, [8].)

What we are going to do in this paper is to show that the "experiences of his scientific career" were mainly his experiences in the study of invariant theory and other algebraic works, and that their influences are even stronger than the impression that comes from Bernays' statements. Such experiences not only caused Hilbert his resistance to Kronecker, but also influenced him technically. Hilbert seems to have built his foundational theories modeled on these early algebraic experiences.

We will show that there is strong conformity between Hilbert's foundational works in 1900's and 1920's, and algebra/arithmetic he was concerned with in 1890's. His axiomatic foundations and formalistic foundations of mathematics by finite axiom systems and finite derivations strongly conform to Kroneckerian finitistic foundations of arithmetical-algebraic mathematics with Modulsysteme. In 1920's, he repeatedly compared his "constructivization" of the proof of Gordan problem with his planned consistency proof in the sense of conservation over the statements admissible in the finite standpoint. (The "constructivization" means that Hilbert gave an algorithm to compute the solution of the problem. According to B. Sturmfels, Hilbert's "algorithm" had a flaw, where he referred to Kronecker's method. The flaw has been fixed by later studies. Sturmfels thinks that the gap is relatively small so that he calls the correct algorithm after Hilbert [7].) We may think that Hilbert's proof theory is an extension of Kroneckerian foundation of mathematics by replacing algebraic formulas with logical formulas, and ideal elements with non-determinate elements (variables). It was of course essentially augmented by the notion of consistency proof which seems overlooked in Kronecker's foundations.

We list more similarities:

1. The completeness axioms in geometry and real numbers and the the notion of completeness of axiomatic or formal systems conform to the notion of full invariant systems.
2. The conviction to completeness of formal systems of arithmetic conforms to his *General Finiteness Theorem* now known as Hilbert's finite basis theorem.
3. Algebraic characters of Hilbert's formulations of logic, e.g., early algebraic formulation of logic in 1900's, and τ or ϵ-calculus in 1920's.
4. The definition of consistency by conservativity over finite statements or impossibility of derivation of $0 = 1$ conforms to the notion of "extension" in algebra.
5. His planned proof of consistency by "try and error" ϵ-substitution method conforms to his proof of *General Finiteness Theorem.*
6. "Efficiency" arguments on assumed decision methods for predicate logic or entire mathematics conform to the efficiency problems in computations in invariant theory.

From these historical facts and conformity between his formalistic foundations and algebraic notions, it is quite plausible that his foundational theories are consciously or unconsciously modeled on his early algebraic works and Kroneckerian finitistic foundations of mathematics.

There are some statements by Hilbert and his school, which look very strange for the contemporary logician who take "computation" in the sense of Turing. For example, as Hao Wang once wrote, why could they believe in the completeness of formal theories of the first order arithmetic, which implies a decision method of arithmetic by Turing's argument? Did they simply overlook the simple argument by Turing? If we stand in the "algebraic" position Hilbert probably stood in at least in 1897, the notion of computation becomes to look more restrictive so that Kleene's μ-operator is not a computation at all. Then, it is even natural to think that formal theories are complete even if decision methods of mathematics in practical sense is illusory.

Today, quite a number of mathematicians think Hilbert program is a wasted effort of a great old man. Even if a mathematician gives respect to the effort to save mathematics, he/she finds that Hilbert went too far from "real mathematics." However, such an impression is not very correct. For Hilbert, his proof theory is a natural continuation of the works of his youth. On success of his proof theory project, he should have given the final answer to Kronecker and Gordan as the last touch on the canvas of his mathematical life. For Hilbert, proof theory was not a marginal work at all, but was very central in his mathematics.

In the rest of the paper, we will discuss details of the points raised above by giving evidences in Hilbert's and other's primary literatures. We owe our basic standpoint taking Hilbert as a revolutionary of the transition from mathematics by computation to mathematics by concept-thinking that took place in 19th century, to Laugwitz[3][1]. A similar standpoint is found in Gray[2]. However, we would be the first to relate this standpoint to his foundational works.

The conformity of Hilbert program and his invariant theory works was pointed out to the first author by S. Kimura. This work was possible only after this illuminating remark.

References

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[1] Laugwitz claims that Riemann is most important to initiate it. We think that Hilbert is even more important from "sociological" point of view. Riemann was the genius creating the magical new method, but it is Hilbert who converted the magic to science. We either disagree with Laugwitz ignorance of Cantor's importance. Since we think his preparation of the materials for Hilbert's science is quite important for "architecture of mathematics" in Bourbaki's sense.

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