Contents

Hilbert's early philosophical thoughts and their influences to his studies of the foundations of mathematics

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1. Introduction

In David Hilbert's *Nachlaß*, there are three mathematical notebooks, written through his academic life. They are also known as Hilbert's mathematical diaries. However, the notebooks consist of sketchy short notes without dates, and so we call them notebooks rather than diaries.

The notebooks are especially rich in their discussions on the foundations of mathematics, by which we can understand the origin and motivations of Hilbert's foundational studies. In this paper, we will report some of our research results on his foundational thoughts up to the early 1890's, when he had finished his study of algebraic invariants and was main studying he algebraic number theory as well as geometry.

It should be noted that the studies presented in this paper are not exhaustive by any means. We will discuss only two specific subjects, the origins of the solvability principle of every mathematical problem, proclaimed in his 1900 Paris lecture, and Hilbert's axiomatics.

There are many other important foundational subjects on which the notebooks contain much important information, e.g., the role of computation in mathematics, and the ontology of the irrational numbers. We will examine some notes on these two subjects, since they are deeply related to our discussions on the problems of solvability and axiomatics. However, they would not be exhaustive in any sense.[1]

Probably, the most interesting note that we have found is a note on p.37 of the first notebook(fig. 3.8), which we will call *the solvability note*. It is estimated to have been written in 1888-9. During that time, Hilbert was developing his revolutionary theory of algebraic forms.

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[1] See Hashimoto's forthcoming paper [18] for the development of Hilbert's ontological thoughts on the irrational numbers based on studies of the mathematical notebooks.

In the note, Hilbert proclaims the solvability of every mathematical problem, which was pronounced to the public about 10 years later in his famous 1900 Paris address. In 1900, Hilbert proclaimed that every mathematical problem is solvable in the sense that either it is solved in the expected way, or the solution in the expected way is proved to be impossible. Brouwer took this assertion as a serious philosophical or scientific assertion, by identifying it with the law of the excluded middle, and rejected both [5]. One of the Brouwer's biographers, van Dalen, expressed his puzzlement at why Brouwer took Hilbert's "motto" as a serious scientific or philosophical statement (p... t[69]). Most contemporary readers would agree with van Dalen.

However, it seems that Brouwer was right. Hilbert was quite serious about his assertion in the solvability note. In the solvability note, Hilbert claimed that every mathematical problem could be reduced to a kind of "normal form" and the problems of the normal form are always solvable by finite steps of operations.

According to the terminology of modern computability theory, the normal form is interpreted to represent the class of the Π^1_1-propositions. Of course, there is *no* decision algorithm for this class, and, thus, it is easy to give counterexamples for the solvability principle from the modern theory of computability.

It seems that young Hilbert had already noticed such difficulties in his claim. He wrote two difficult examples for his assertion on solvability; one such example is deciding whether 7777777777 appears in the decimal expansion of π and another is squaring the circle. It is interesting that the first of the two examples of "difficulties" is essentially the same example as the "counterexample" to the law of the excluded middle presented by Hilbert's adversary L.E.J. Brouwer about 35 years later (p. 270, [4], see 10.4 in [69], [70]). Although it has not been proven yet, the problem of deciding whether a given finite sequence of digits appears in the decimal expansion of π is believed to be undecidable. If it is so, it gives a simple concrete counterexample to the solvability principle as Brouwer argued.

Hilbert was, however, determined to believe the solvability principle, even though he had noticed the difficulties in his claim. It is likely that he intended even to *prove* the solvability of mathematics in some way. At least, he maintained the solvability principle as an axiom of human reason, as he called it "the axiom of solvability of every (mathematical) problem" in the Paris lecture.

To our knowledge, the note is the earliest writing in which Hilbert firmly asserts the "completeness" of mathematics. However, it is not the first time that he thought about the problem. There are earlier notes in which the problems of completeness and consistency of mathematics are vaguely discussed. In one such note, he related the problem to Kant's philosophy.

The readers might think that these philosophical thoughts transitorily occurred in the great mathematician's mind only in his youth and evaporated as he grew up into a leading mathematician. However, that is not the case. There is a great deal of textual evidence of influences of these thoughts throughout his academic life and even to the time of his proof theory in the 1920's.

For contemporary readers, it might be unbelievable that the great mathematician who opened the door to 20th century modern mathematics formulated and believed such a transcendental hypothesis. In the latter half of the 19th century, some modern skeptics of science, such as Emile du Bois Reymond's "ignoramus et ignorabimus", had already prevailed in German society. At the very least, it is quite puzzling that Hilbert believed in the existence of a finite procedure which would trivialize his beloved mathematics by solving all mathematical problems. However, we will argue that Hilbert had a philosophical ground on which to hold the solvability principle without making mathematics trivial, basing our discussion on some evidences from his lectures, notes, and published papers.

Through a careful textual analysis, we have dated the solvability note. It was written after his non-constructive solution of Gordan's problem through the finite basis theorem, and before his "constructive" solution of the same problem by Nullstellenstaz. Although there is no direct textual evidence on the relationships between his invariant theory and the solvability principle, we will point out that there are many circumstantial evidences of such a relationship.

Besides the discussions on the solvability principle, we will also discuss the origin of Hilbert's axiomatics. Corry has pointed out the influence of Hertz's thoughts on rational mechanics on Hilbert's axiomatics [7]. A note in the second notebook strongly supports Corry's theory. In the note, Hilbert wrote about a research plan of axiomatics of mechanics and some other areas of physics, and then wrote: "do it first for geometry." In a note written soon after the note on axiomatics of physics, he wrote about the consistency proof of the axioms of geometry by means of the model of analytic geometry. In the next note, he gave his famous definition of "existence" of mathematical objects by consistency.

The present paper consists of four sections and an appendix. Section 1 is this introduction. In Section 2, we survey Hilbert's mathematical works up to 1897. The notes which we have studied are considered to have been written during this period. This serves as a background for our research. In Section 3, we will present the research results using textual philology. In Section 4, we will present our interpretations of the contents presented in the previous section. The results in this section are more or less *hermeneutic*. The discussions in sections 3 and 4 are closely related, but we presented them separately to distinguish the highly objective results in the former section and the more subjective but more informative interpretations in the latter section. In the appendix, we will give some philological remarks.

Some of the materials of the present paper have already been made publicly available in the annotation of our book [24] and a web site [23]. Since the book is meant for the general public, detailed academic discussions are not included there.

2. Hilbert's research up to the era of geometry

In this section, we will survey Hilbert's mathematical researches from 1885 up to 1899, when Hilbert published his celebrated volume of the foundations of geometry. We will focus on the period 1888-1892, since most of the important notes, which we will present below, were written during this period. The survey is written mainly based on known and well-established interpretations of history without mentioning our findings on the three notebooks. Thus, it will serve as a background and provide "reference points" for the description of our research which will be presented from the next section.

2.1. Theory of algebraic invariants 1885-1892

The theory of algebraic invariants was Hilbert's first research area. He studied mathematics at Königsberg University in his hometown, and obtained a Degree of Philosophy in 1885 with his study on algebraic invariants. After his doctoral thesis, he continued to study algebraic invariants till 1892. His research activities were especially intense and novel during the period of 1888-1892. In this period, he solved so many important problems of the area that finally he felt that he had solved all the important problems of the area and nothing further was left for him to do. In the autumn of 1892, he wrote to his friend Minkowski that he would leave the area and move to number theory [3, 61]. His novel non-constructive research paradigm invented in the couse of studying algebra was to be generalized and polished by his students and followers, especially by Emmy Nöther, so that it became the paradigm of 20th century algebra.

Hilbert is regarded as the man who championed the non-constructive mathematics of Dedekind and Cantor and turned it into the main stream of 20th century mathematics. However, in the very beginning of his career Hilbert was still a standard 19th century algebraist who did mathematics through formula computations rather than non-constructive thinking.

A sign of a change came in the winter semester of 1886/7. He was 24 years old and a Privatdozent at Königsberg University. At this time he gave a lecture course on algebraic invariants, which challenged Gordan's problem, a grand open problem of the time.

Gordan's problem was to find a finite "basis" in an infinite set of forms (homogeneous polynomials) whose elements are called invariants of a given form (or a set of forms). The problem itself is originally conceived of by the renowned British mathematician Cayley and was solved positively for binary forms by the German mathematician Paul Gordan. After producing the solution, Gordan was called the king of invariant theory, and the problem of the general case was called Gordan's problem.

In 1887 lecture course on algebraic invariants, Hilbert discussed the problem (Cod. Ms. Hilbert 521 pp.193-194) as follows: Gordan's method is too complicated for actual computation and can be carried out only for small cases. The flood of the papers on the subject are no more than practices of calculation. The important and fundamental problem is only the finiteness of the system (see 4.4 for a detailed account).

The writing shows that he began to think about not only the mathematical problem itself, but also the framework in which it was to be solved. In Section 3, we will examine several philosophical notes, which are likely to have been written in this period. They will show that Hilbert was pondering on the very foundations of mathematics from a rather early time, although he did not hint at it in published papers. There was an unspoken agreement among mathematicians, at least German algebraists, that the problem, having been open over two decades, should be solved by giving an algorithm to compute the basis from the given forms. However, Hilbert began to unbind the rope which had been tacitly tying the mathematicians up.

From March 9th to April 7th 1888, he made a grand trip to visit 21 prominent mathematicians, including Kronecker, Gordan and Klein. He sent some letters to Klein on the trip (cf. [13]) and made notes of the visits to these mathematicians. On March 16th, Hilbert wrote to Klein from Leipzig that Gordan was kindly coming to see him in Leipzig. At that time Gordan was at Erlangen University, where Klein used to be his college and co-worker, and they had been friends since then. Hilbert wrote about Berlin mathematicians whom he had visited, especially on Kronecker. Kronecker was unexpectedly friendly to him and they talked longer than had been planned, mainly on the foundations of mathematics. Hilbert took a note of Kronecker's way to introduce the irrational number $\sqrt{5}$ as the indeterminee x modulo $x^2 = 5$, which appeared in his lectures [50, 51].

On March 21st, he wrote to Klein again from Leipzig. This time, he reported that he had found a short new proof of Gordan's theorem of the finiteness of binary invariants with the help of Prof. Gordan. Hilbert compiled a short paper on the proof [29] and submitted it to Mathematische Annalen when he visited Klein in Göttingen on the way back to Königsberg.

Although the new proof is really short, it belongs to the old regime. His new proof is based on a well-known theorem on homogeneous Diophantine equations, for which Hilbert cited Gordan's textbook on algebraic invariants.

A breakthrough came very soon then thereafter. On the 3rd of October 1888, Hilbert sent a letter and a manuscript of a short paper to Klein for publication in Göttinger Nachrichten. In the paper, Hilbert reported the finite basis theorem without proof, and applied it to solve Gordan's theorem in the general case. He then published in succession another two papers under the same title [30].

The full paper combining the three short reports with full proofs appeared in Mathematische Annlen 1890 [31]. Its manuscript was sent on February 15th 1890 to Felix Klein, then the editor-in-chief of the journal. Klein asked his friend Gordan his opinions on Hilbert's paper. Gordan quickly replied with a letter dating on February 24th 1890 [13]. He wrote that it was a fine first step to the solution of the problem but the paper was not adequate to be published in the journal (see p.65 of [13]). The ground of rejection was the fact that Hilbert's proof lacked of Eintheilungsprinzip (classification principle), which, Gordan claimed, should be present in any Rekursionsbeweise (recursive proof). Eintheilungsprinzip would mean a well-founded relation in modern terminology. Gordan assumed that, in a proof by Rekursionsbeweise, a simple Eintheilung of all algebraic forms must be given first, and then that the proof must proceed from simpler forms to more complicated forms. However, there was no such Eintheilungprinzip in Hilbert's proof.

Klein reported in [49] that Gordan had said to him that "it is not mathematics but theology," when Klein tried to call Hilbert to Göttingen as a professor after his appreciation of the work.[2]

Hilbert's 1890 proof of the generalized Gordan theorem comprises two steps. The first step is Hilbert's finite basis theorem and its application to the set of invariants of a given ground-form. The finite basis theorem is a general theorem that can be stated as follows. For any sequence F_1, $F_2, \ldots$ of forms, there is a number m satisfying that, for any i, there are forms A_1, $A_2, \ldots, A_m$ so that the linear combination $A_1F_1+\cdots+A_mF_m$ equals to the given F_i. Using the theorem, Hilbert showed that there are finite number of invariants i_1, $i_2, \ldots, i_m$ so that all the invariants could be represented as the linear combinations $A_1i_1+\cdots+A_mi_m$, where A_i are some appropriate forms. In the second step, using Ω-process, a symbolic process originally conceived of by Gordan and Mertens, $A_1, \ldots, A_m$ are replaced by invariants simpler than the given invariant. By applying the same process (step 1 followed by step 2) to the simpler invariants, we can gradually turn the linear forms $A_1i_1 + \cdots + A_mi_m$ into some polynomials of i_1, $i_2, \ldots, i_m$.

In his letter to Klein, Gordan did not mention anything on the second step. His accusation of "Theology" was directed to be the first step of the proof. The finite basis theorem is non-constructive[3], namely, it does not give any algorithm to compute m from a given sequence F_1, $F_2, \ldots$.[4] It was thus impossible to prove the theorem in the way that Gordan had expected according to Hilbert's method.

It is not known when Gordan noticed the "defect" of Hilbert's proof for the first time. Hilbert would have sent his first paper in Göttinger Nachrichten to Gordan in the end of 1888. Gordan sent Hilbert a letter dating December 26th 1888 and it is a reply to Hilbert's paper sent to Gordan. Although Gordan did not mention the name of the paper, it would be the first of the three Nachrichten papers. See B for a transcription and a translation.

Since Hilbert's paper did not include any hint of the proof of the finite basis theorem, Gordan did not see Hilbert's "theology" then and congratulated Hilbert on his great achievement. Nonetheless, Gordan had already sensed some difficulty. He wrote his anxiety on the finite basis theorem and suggested restricting the finite basis theorem to concrete cases, although Hilbert's proof of Gordan problem cannot be constructivised in this manner.

German contemporary algebraists such as Gordan believed that algebra should be done by algorithmic manipulations of symbols: see e.g. I, 2 of [12], 4 of [52]. It is well-known that Kronecker had an especially strong opinion on this

[2]He tried to call Hilbert twice in 1892 and 1895. Gordan's utterance seems to have been made in 1892.

[3]It is well-known that there are several different versions of "constructiveness" of proofs. In this paper, the word "constructive" is used in a rather relaxed fashion. In almost all cases, it means that a proof contains some information to compute solutions. But, in some cases, it may mean that the proof employs only intuitionistic logic. The latter sense is narrower than the former sense.

[4]In the terminology of recursion theory, a sequence is an oracle. In computer technology jargon, it is called a stream. For example, virtually non-interrupted incoming information from a network line is a stream for a PC connected to a network. For a mathematical explanation of the non-computability, see e.g. [25].

point. He claimed that the whole pure mathematics including analysis should be done in this way and would be done so in future [50, 51]. It is understandable that some mathematicians had difficulty accepting Hilbert's proof of the finite basis theorem in view of the image of mathematics which was dominant in that period.

Franz Meyer, a notable algebraist of that era, compiled some 200 pages long report on invariant theory for Deutschen Mathematiker-Vereinigung (DMV) [55] in 1892.[5] The report is said to have been very influential and it was translated into several languages. Although Meyer in his report described Hilbert's work as a recent important work, he cautioned that the finite basis theorem conflicted with the foundational problem as it used an *arbitrary* (willkürlich) sequence, which amounted to Cantor's theory of irrational numbers defined in terms of the fundamental sequences. He mentioned a tendency of scientific thinking which would reject such an infinite process (footnotes, p.144 [55]).

Despite such initial oppositions, Hilbert's new algebra was to be accepted relatively quickly. For example, Gordan changed his mind soon. Later, he even contributed to the new method by improving the proof of the finite basis theorem. Klein reports that he said "I convinced myself that theology has also its merits."

Nonetheless, Hilbert himself was not completely satisfied with the non-constructive solution. He was seeking for a new method of solving the same problem by which one could compute the bases. He achieved this goal by an entirely different method based on a general theorem now known as Nullstellen-zsatz. He conceived of the theorem as a generalization of Max Nöther's famous theorem on the intersection of two 2-dimensional algebraic varieties and called it the general Nöther theorem or the general Nöther principle in his mathematical notebooks.

The new principle together with the finite basis theorem became a basic and general tool of modern algebra. However, according to his writings, giving a constructive alternative to his non-constructive theory seems to be one of the main motivations for his Nullstellensatz.[6] Klein made clear this point, when he received Hilbert's manuscripts of his article including the constructive proof for publication. He congratulated Hilbert on his discovery of a method which invalidated Gordan's criticism.

Hilbert published the new solution starting with three short papers in Göttinger Nachrichten just as in the case of the first solution [32]. The three short reports were published in 1891 and 1892, and the full final version "Ueber die vollen Invariantensysteme" was published in 1893 in Mathematische Annalen.

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[5] One of the tasks of DMV, which had been founded only a year before, was to publish comprehensive surveys of major areas of mathematics. Hilbert's Zahlbericht was an example and Meyer's survey was another example.

[6] We do not mean that it is *the* motivation. Some evidence from the first notebook suggest that he may have been seeking the principle from rather early time. His search might already have started before his discovery of the finite basis theorem. We thank Shuichi Kimura for pointing us the possibility.

The paper was submitted on September 29th 1892. Blumenthal [3] reports that Hilbert simultaneously (gleichzeitig) wrote to Minkowski and expressed his intention to depart from the theory of algebraic invariants.[7]

2.2. Theory of algebraic number fields 1892-1897 and Geometry 1891-1899

Hilbert mainly studied algebraic number theory from that time until the spring of 1897. The final outcome of this study is now known as "Zahlbericht", the number theory report. In the report, he restructured the algebraic number theory developed by Kummer and Kronecker, which were mainly based on formula computations, using a new theory based on Dedekind's non-constructive ideal theory. In the introduction of the report, dated April 10th, 1897, he praised Kummer's theory of higher reciprocity laws, but he declared to replace Kummer's computational mathematics with Riemann's mathematics by thinking. He characterized his improvements of Kummer's theory of higher reciprocity laws as follows: "I have tried to avoid Kummer's elaborate computational machinery, so that here Riemann's principle may be realized and the proofs completed not by calculations but purely by ideas."[8]

From 1892 through 1897, Hilbert extensively studied the theory of algebraic number fields. In the period of 1892-1897, Hilbert published almost exclusively on algebra and algebraic number theory. However, they were not the unique areas on which he worked during this period. This was also the time when his study of the foundations of geometry was initiated and developed. The details of the history of his study of geometry are now available in works by Toepell [66] and by Hallet and Majer [15].

According to the list of Hilbert's lecture notes in Hallet and Majer's volume of Hilbert's lectures on the foundations of geometry p.609-623 [15], Hilbert's first lecture on geometry was focused on the descriptive geometry to be delivered in the summer semester of 1888. this lecture is thought to have been announced but later canceled. Hilbert's first *given* lecture course on geometry seems to be the one on projective geometry he conducted in the summer semester of 1891.

In September of the same year, Hilbert visited Halle University to attend the first annual meeting of DMV, which was held during 22-26th September. He gave a talk there on his theory of algebraic invariants. At the same meeting, a Privatdozent Hermann Wiener of Halle University gave a talk on the foundations of geometry "Ueber Grundlagen und Aufbau der Geometrie", pp.45-48, Jahrbericht der Deutschen Mathematiker-Vereingung, Band 1, 1890-91.

It is widely believed that the fundamental ideas of Hilbert's axiomatics were inspired by the talk. Hilbert's student Otto Blumenthal described Wiener's influence on Hilbert in his Lebensgeschichte for Hilbert[3]. Blumenthal reported a now famous story of geometry of tables, chairs and beer mugs. He wrote that,

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[7] Hilbert's letters to Minkowski have unfortunately been lost. It is said that Blumenthal had read them and later wrote Hilbert's Lebensgeschichte [3]. References to Hilbert's letters in our article are taken from [3], as in the case of other documents on Hilbert, e.g. the biography by Reid.

[8] The passage is quoted from the English translation of [34].

in a waiting room of Berlin station on the way back to Königberg from Halle, Hilbert said to fellow geometers: Man mußjederzeit an Stelle von "Punkte, Geraden, Ebenen" "Tische, Stühle, Bierseidel" sagen können.

The outcome of this inspiration was presented in the lecture course on the foundations of geometry in the 1894 summer semester. It is said that it had been planned to be given in 1893, but was canceled, as there were few registrations. Its lecture notes, "Grundlagen der Geometrie", Cod. Ms. D. Hilbert 541, has been published as Chapter 2 of [15]. The dependency/independency-analysis between propositions, the most important feature of Hilbert axiomatics from mathematical point of view, was not pursued enough at that time. However, already meta-mathematical views have unclear emerged in the lecture notes (see Majer's introduction of Chapter 2, [15]).

For example, the idea of "existence=consistency" already appears there though in a vague expression. Hilbert discussed "Möglichkeit" of Euclidian and no-Euclidian geometries in terms of consistency. It was in the 1898/99 winter semester lecture notes "Grundlagen der Euklidischen Geometrie", Cod. Ms. D. Hilbert 551 that for the first time a clear and literal definition of "Existenz" in terms of consistency was given (consult footnote 173, p.282 [15]). In the same year, the celebrated book "Grundlagen der Geometrie" was written based on the lecture notes as the first part of the Festschrift for the erection of the Gauß-Weber monument in Göttingen. In the lecture course and the book, a complete body of Hilbert's axioms was presented for the first time.

We now close our brief survey of Hilbert's mathematical career up to the end of 19th century. In the following sections, we will present some of the important notes from the notebooks which are considered to be written in the course of Hilbert's career up to around 1894 and estimate the dates of their writings by comparing them with the history explained in this section.

3. Early notes on foundational issues

In this section, we will examine the notes on foundational issues from Hilbert's first and second notebooks. We have noted in the introduction that we will sometimes speculate on the meanings of notes based on our knowledge in mathematics or philosophy. In this section, we will, in principle, avoid such speculations, and postpone such discussions in the next section, where we will try to draw deeper conclusions by free thinking from some speculative yet convincing hypotheses. This is to distinguish these two kinds of discussions as far as possible. However, it will be of course impossible to make them distinguished completely, since any philological reading may include some amount of interpretations. We will also violate the principle from time to time for readability, e.g. the discussions in 3.13.

Let us illustrate our "objective philological readings" in this section with an example. We will maintain that Hilbert started to reflect on the solvability of every mathematical problem between March 1888 and July 1891. This conclusion can be drawn only through textual analysis. For instance, the lower bound of March 1888 was inferred from the finding of a note which contains some information on a mathematical theorem, which was obtained between March and September of 1888.[9] All the notes after it are thus estimated to have been written after March 1888.

In the next section, we will not hesitate to resort to more speculative methods to draw deeper conclusions on this issue. We will maintain that his finite basis theorem might have inspired the solvability principle by analyzing his later writings on the theorem and the logical similarity of the principle and the theorem.

Even for the philological analysis of this section, some assumptions on the texts are inevitable, such as "the notes are lined up in a chronological order", and we also need to index the notes for systematical reference. These philological matters are presented in the appendix and are not discussed here. In the rest of this paper, we use indexes of the index system explained in the appendix, e.g. Book1, Page37, Region2. The readers are kindly advised to have a glance at the appendix before proceeding to the subsections below.

3.1. The first notebook

On the front cover of the first notebook, it is written in Hilbert's handwriting that it was acquired during the winter of 1885-1886 in Leipzig. The second notebook is dated 12 September 1892, apparently with his wife Käthe Hilbert's handwriting. Thus we may assume that the first notebook, which we will refer to as Book1, had been written until September 1892. Thus the first notebook Book1, which has 133 pages, amounts to having been written for 6 or 7 years.

Book1 and the other two notebooks, Book2 and Book3, are collections of short notes which are almost always separated by short horizontal lines as in Fig. 1. The main tasks of our philological analysis are to estimate the time of writing and to account for the meaning of each note.

Hilbert sometimes referred to published literature, whose exact years of publication are known. Such references are the most exact and important evidence for the estimation of the time of writing of a note. For example, the second note in p.6 (the note Book1, Page6, Region2 according to our index system) refers to a formula (5) in his "note" (meine Note) "Ueber d. Diskriminatenfläche". The title of the reference is not completely written, but there is a unique paper which includes the term "Diskriminatenfläche" in the title, viz. "Ueber die Singularitaeten der Diskriminatenflaeche, Mathematisches Annalen Bd. 30, pp.437-441, 1887". The paper is submitted on June 2nd 1887. The paper consists only of 5 pages and is suitable to be called a note. The note of Book1, Page6, Region2 is a memorandum on how to compute the value of the formula (5) in the aforementioned paper. It is therefore quite likely that the note was written after completion of the paper or even after its publication. Thus, it would be appropriate to infer that the notes following Book1, Page6, Region2 were written after the spring of 1887. This context shows that Hilbert did not write many notes

[9] The estimation of the period of obtaining the theorem has been made on the ground of the correspondences between Hilbert and Klein.

from the winter of 1885/1886 through early 1887. This notebook has 133 pages and yet had been written over the period of six or seven years. It means that the average of the pages per year is about 20 pages. There are no interesting notes up to Page6. The situation changes in Page 7, where he mentioned the 10th problem

3.2. Hilbert's tenth problem

In Book1, Page7, Region5 Hilbert clearly formulated the 10th problem among his famous 23 problems in mathematics pronounced in 1900 for the first time.

Book1, Page7, Region5
Transcription: Beweisen, dass man durch eine endliche Anzahl von Operationen entscheiden kann, ob und wie viele ganzzahlige Losungen ϕ(x,y...)=0 mit ganzzahlige Coeffi⟨⟨ cienten ⟩⟩ ⟨Koeffizienten⟩ be⟨⟨ sitzt ⟩⟩.
Translation: Prove that one can decide through a finite number of operations if ϕ(x,y...)=0 with integral coefficients possesses integral solutions and how many integral solutions it has.

This note might have been meant to be purely mathematical and unrelated to the foundational issues. However, if one accepts Kronecker's view of pure mathematics (cf. 20 D in [10]), every problem of pure mathematics is a problem of Diophantine arithmetic, and deciding its validity is identical to the tenth problem. We have no clue to Hilbert's intention of the note at this moment. We will call this note, but it might have some foundational meaning as explained in 4.1 below.

The next note to the tenth problem note concerns a plan of his lecture on descriptive geometry (Colleg über Darstellende Geometrie). The lecture is believed to have been scheduled for the winter semester of 1888, but was not taught then. See p. 14 [66] and the list of Hilbert's lectures compiled by Hallet and Majer [15]. Thus it is quite likely that this note was written in 1887 or early 1888. It is consistent with our estimation of the writing time of Book1, Page6, Region2 given above.

3.3. On the existence of irrational numbers

If the tenth problem note was written without foundational intention, perhaps Book1, Page8, Region1 is the first note of such a kind.

Book1, Page8, Region1
Transcription: Es ist völlig willkürlich bei den gebrochenen, rationalen Zahlen abzuschiessen[abzuschliessen?] und die irrationalen Zahlen nicht mehr zuzulassen. Wir können auch die rationalen gebrochenen Zahlen nicht mehr als Cardinalzahlen auffassen sondern müssten dann[?] logischer Weise [logisherweise] bei den ganzen[underlined] positiven[underlined] Zahlen (vielleicht streng bloss von 1 bis 5 höchste[höchstens] 7)stehen bleiben. Schon die Zahl 1/2 erfordert die Anschauung der Strecke, wir können die Mitte einer Strecke praktisch nie genau angeben, wohl aber genau anschauen und vorstellen. Eben so genau können

wir auch den Punkt $\sqrt{2}$ anschauen und vorstellen. Es giebt ⟨gibt⟩ ein Ueberschreiten[Überschreiten] vom rein anschaulichen zum rein formal gedachten[?]. Diesen Uebergang[=Übergang] vermittelt[?] ein breiter Weg mit vielen gleich bequemen Zugängen und offenstehenden Thoren[=Toren], während Kronecker sich darin verhornt[?] hat durch ein wilkürlich gelegte Hemmnisse[=Hemniss] durch ein Nadelöhr durchzuzwängen.
Translation: It is completely arbitrary to accept only fractional, rational numbers and not to admit irrational numbers. We cannot regard rational numbers as cardinal numbers anymore, but, then logically, we had to stop at the positive integers (perhaps strictly only from 1 to 5 at most[??] 7). Although the number 1/2 requires the intuition of a segment and practically we can never point out the center of a segment, we can surely examine and show it. Just as exactly as that, we also examine and show the point????. There is transgression from pure intuitions to purely formal thoughts. This transition takes us to a wider path which connects to many doorways that open easily and gateways standing open, while Kronecker has to squeeze through a needle's eye due to an arbitrarily laid hindrance.

We will call the note *the first irrational numbers note*. Note that Hilbert strongly argued against Kroncker is prohibition of the concept of irrational numbers on a philosophical basis. In 1904 talk on the foundation of arithmetics [39], Hilbert criticizes Kronecker's foundational thoughts in a similar vein. In the talk, Hilbert called Kronecker a dogmatist, since "he accepts the integer with its essential properties as a dogma and does not look further back" (English translation is from [26]). Hilbert emphasized the necessity of a foundational analysis of the notion of integer, and gave his first attempt of the consistency of the theory of the integer arithmetic.

After the estimation of the writing times of Book1, Page6, Region2 and the note on geometry lecture, we estimate the time of writing the note on irrational number is around 1887-1888. Of course, it is possible that Hilbert wrote on the geometry lecture plan and did not write for a long while. However, there is a note estimated to have been written in 1888 on Page33. Thus, it is rather sure that it was written till 1888.

Regardless of any interpretative questions regarding Hilbert's intentions in his of the note on the irrational numbers, the note clearly shows that Hilbert started to ponder the foundational issues by the end of 1880's. In that time, his main or virtually unique area of research was the theory of algebraic invariants. The theory of irrational numbers are practically irrelevant to the theory of algebraic invariants. Thus, it is likely that Hilbert's interests are of a general character.

3.4. Solution of Gordan problem?

Unfortunately, we have not found a certain place in Book1 that corresponds to the late spring and summer of 1888 when Hilbert invented the finite basis theorem and solved the Gordan problem.

However, there are at least two notes which seem to be related to the problem and suggest that Hilbert had solved the problem when he was writing them.

Book1, Page13, Region5 Auch unendlich viele Formen haben nur ein endliches System von Invarianten.
Translation: Also an infinite number of forms have only a finite system of invariants.

Book1, Page14, Region3 Beweis dass durch für alle Untergruppe der continuierliche [=kontinuierliche] lineare Transformation die Zahl ale=[alle] Invarianten endlich ist[,] z.B. wo eine quadratiere Form in sich übergeht. Beweis durch Aufstellung der entscprchenden Differentialgl.
Translation: Proof that the number of invariants is finite for all subgroups of continuous linear transformations, e.g., where a quadratic form turns in itself. Proof through a setting of the appropriate differential equation.

In 1890 Mathematische Annalen paper and its early reports in 1888-1889, Hilbert considered extensions or refinements of the general Gordan theorem. Since his proof was simple and essential, such applications were easy. These two notes give such extensions.

It is not clear what Hilbert meant by the first note. If it means that the set of invariants of infinite many ground forms have finite full invariant systems, then it is simply false. Since this extension is not mentioned in his published papers and lecture notes, it might be a mistake. However, it might suggest that he already had a non-constructive proof for the Gordan theorem.

In the second note, Hilbert restricted transformation groups by which forms are transformed. If we replace the system of all the linear transformations in the definition of invariants by its subgroup, we have a new notion of algebraic invariants. Does the Gordan theorem remain valid for such new invariants? This was the extension, which was later presented as Hilbert's 14th problem of his 23 problems.

In the third report in Göttinger Nachrichten 1889 [30], Hilbert presented such an extension by mentioning it as a question in the direction of Klein and Lie. He gave there two examples of subgroups, and one of them is a subgroup of linear transformations preserving a given quadratic form, which is exactly the example in the second note given above.

Hilbert did not make clear if forms and invariants in these notes refer to forms and invariants of n-variables or only binary forms. As we noted in the previous section, Hilbert had had another new proof of the binary case which does not use the finite basis theorem right before he found it. Thus, he may be referring to the older proof rather than his proof appeared in 1890. However, this does not seem probable, since invariants relative to subgroups of linear transformations are not mentioned in the 1889 paper of the older proof.

On this philological evidence, we maintain that Hilbert had already obtained his solution of Gordan problem by means of the finite basis theorem when he was writing the two notes in Page13 and Page14.

As we mentioned earlier, Hilbert did not write many notes in the middle of 1880's. The situation was much changed after 1888. He wrote many notes on various subjects. Perhaps, there was a kind of explosion of ability and many ideas flowed out without interruption. They were not only mathematical ideas, but some philosophical ideas came to him, as well.

3.5. Definitions never fail

It is the beginning of 1890's when Hilbert conceived the idea of "existence=consistency", see 3.15. However, we may find a germ of his idea might be seen in the following note.

Book1, Page18, Region4 Eine Definition kann niemals "falsch" sein auch kann sie niemals unlogisch sein. Denn sie ist etwas rein sprachliches. Demgemäss kann sie unverständlich sein oder mehrfacher Interpretation fähig. Nur in diesem Falle ist sie ohne weiteres fort zu werfen und unbrauchbar. Ausserdem kann das zu definirende[=definierende] Wort sich mit dem sonst üblichen Gebrauch ⟨ins⟩ Widerspruch befinden. Dies ist aber kein wesesentlicher[=wesentlicher] Uebelstand[=Übelstand].
Translation: A definition can never be false, or can never illogical, since it is somehow a pure linguistic being. Accordingly they can be non-understandable or allow multiple interpretations. In those cases, it is only that it is readily thrown away or unuseful. Besides them, it is possible that the defining words could be self-contradictory with otherwise normal methods. That is however no essential nuisance.

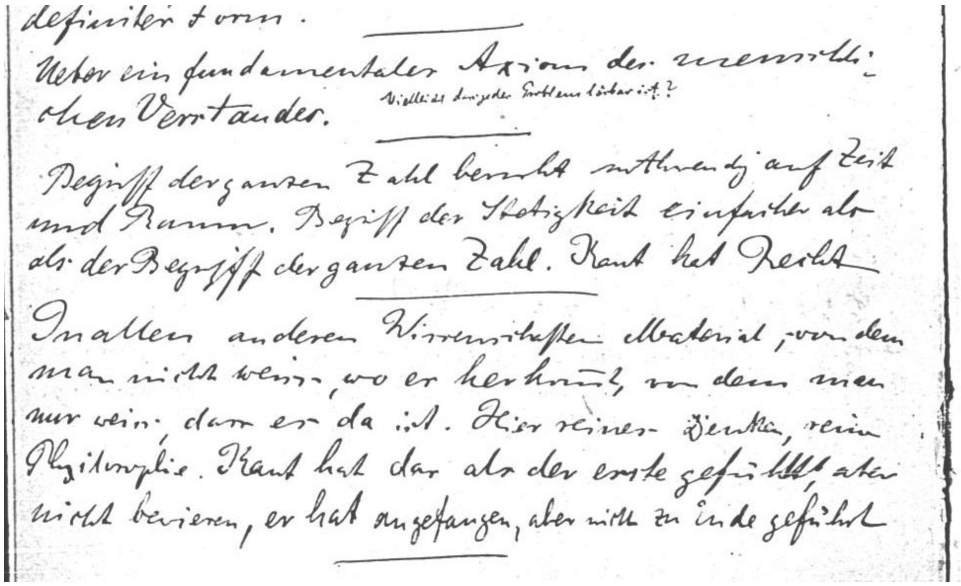
3.6. Notes on Kantian philosophy and mathematics

In Book1, Page28, Region4-6, he wrote three notes related to Kant. Transcriptions and translations are presented below together with their original images in Fig. 1. We will call these three notes *Kantian philosophy notes.*

First note: Book1, Page28, Region4
Transcription: Ueber ein fundamentales Axiom des menschlichen Verstandes. Vielleicht dass jedes Problem lösbar ist?
Translation: About a fundamental axiom of human intellects.[10] Probably ⟨the axiom is⟩ that every problem is solvable?

Second note: Book1, Page28, Region5
Transcription: Begriff der ganzen Zahl beruht nothwendig ⟨notwendig⟩ auf Zeit und Raum. Begriff der Stetigkeit einfacher als als der Begriff der ganzen Zahl. Kant hat Recht. ⟨Two consecutive "als"s must be a mistake.⟩

[10] In the translation [10] of Hilbert's 1900 Paris talk, "Verstand" is translated into "reason". "Verstand" and "Vernunft" of Kantian philosophy are normally translated into "intellect" and "reason", respectively. We translated "Verstand" in the notes quoted in this paper into "intellect". It seems that Hilbert's "Verstand" is closed to "reasoning" in the sense of "mathematical reasoning" or "logical deductions". However, in the most notes quoted in this paper, "Verstand" is used as "menschenlichen Verstande", and we translated them into "human intellects", since it looks more suitable than "human reason".



definiter Form.

Ueber ein fundamentaler Axiom der menschlichen Verstandes. Vielleicht dass jedes Problem lösbar ist?

Begriff der ganzen Zahl beruht nothwendig auf Zeit und Raum. Begriff der Stetigkeit einfacher als als der Begriff der ganzen Zahl. Kant hat Recht

In allen anderen Wissenschaften Material, von dem man nicht weiss, wo es herkommt, von dem man nur weiss, dass es da ist. Hier reines Denken, reine Philosophie. Kant hat das als der erste gefühlt, aber nicht bewiesen, er hat angefangen, aber nicht zu Ende geführt

Figure 1: Kantian philosophy notes: Book1, Page28, Region4-6

Translation: The notion of integers is necessarily based on time and space. The notion of continuity ⟨is⟩ simpler than the concept of integers. Kant is right.

Third note: Book1, Page28, Region6

Transcription: In allen anderen Wissenschaften⟨,⟩ Material, von dem man nicht weiss, wo es herkommt, von dem man nur weiss, dass es da ist. Hier reines Denken, reine Philosophie. Kant hat das als der erste gefu⟨⟨ hlt ⟩⟩, aber nicht bewiesen, er hat angefangen, aber nicht zu Ende gefuhrt.

Translation: In all other sciences, materials, of which one does not know where it comes from, and of which one only knows that it is there. Here ⟨are⟩ pure thinking, pure philosophy. Kant sensed it as the first one, but did not prove it. He began, but did not go to the end.

The last passage of the first note in the small letters are likely added later. He is asking if the solvability of every problem could be an axiom of the human intellect. The time of writing these three notes are estimated around 1888-89. More precise estimation is "after the solution of Gordan problem, which is in between of March and September 1888, and before February 15th 1890". February 15th 1890 is the submission date of 1890 Mathematische Annalen paper.

The ground of the time estimation is the note consisting of Book1, Page33, Region3 and Book1, Page34, Region1. This note includes two examples of subgroups of Hilbert's 14th problem discussed in Book1, Page14, Region3. The note includes mathematical formulas representing two examples of subgroups, and differential formulas necessary to solve Gordan problem for the cases. The formulas are identical to the ones that appeared in the 1890 Mathematische Annalen papers except for the difference of names of variables in the formulas. Thus, it is quite likely that Hilbert wrote down the formulas when he was writing the paper or a little bit earlier. Thus, all the notes appearing before Book1, Page33, Region3 would be written before February 15th 1890.

However, there is another possibility. Although no formulas were presented, Hilbert gave the same example in the third report in the Göttinger Nachrichten, which was submitted on June 30th 1889. Thus this note may be written before then. However, it is possible that the paper was revised after the submission and the part mentioned above was added after the submission.[11]

However, the more conservative estimation of Feb. 15th 1890 could be kept, even if the statements on the example was added after the submission of the paper of Nächricten. Anyway, it will be certain that the notes above were written when Hilbert was working on his new theory of algebraic forms published in 1890.

3.7. The early formalistic notes

The two notes below are Hilbert's very early formalist questions on decidability and completeness of mathematics. It is very likely that they represent Hilbert's very first thoughts of such a kind. We will call them *the early formalistic notes*.

Hilbert did not make use of these later terminologies, but it will be clear that the questions raised in these two notes are related to the later formalistic questions, such as a decision method of mathematics and completeness of mathematics. The last sentence of the second note even reminds us Gödel's incompleteness theorem, although Hilbert tended to deny the possibility of such a theorem.

In the first Kantian philosophy note, he considered the axiom of human intellects. He seems to try to answering his own question in these notes based on his understanding of Kantian philosophy. It is interesting that he thought space and time are more basic and simple than the notion of integers. This thought conforms to his thought in the first irrational number note. Then, he criticized Kronecker's integer-based foundation by regarding numbers as points on the linear continuum.

In the third Kantian philosophy note, he wrote that Kant did not go far enough. In the end of the first note of the early formalistic notes, he is asking about the possibility of *proving* that every cognition must be possible. In the note of the next subsection, we will see his optimism culminate even in a semi-mathematical formulation of the "completeness" of human intellects in mathematics.

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[11] In the present, revision dates are indicated in papers on publication. This is unlikely to have been done in Germany of the time. However, as far as we know, there are no papers with revision date indications in the two journals....

????? When revision indication started?

First note: Book1, Page32, Region1
Transcription: Vorausgesetzt, dass es keine praktischen Schwierigkeiten gäbe, ist der menschliche Verstand vermogend, alle Fragen, die er sich selbst stellt, zu losen? Muss es moglich sein zu entscheiden uber die Quadratur des Kreises, Giebt ⟨Gibt⟩ es eine wohl definirte ⟨definierte⟩, auf die sinnlichen Dinge bezugliche Frage, welche eine Antwort haben muss und die gleichwohl sich nicht beantworten lasst.⟨?⟩ Kann man einen allgemeinen zwingenden Beweis fuhren, dass jede Erkenntniss ⟨⟨ möglich ⟩⟩ sein muss?
Translation: Assuming that there were no practical difficulties, is the human intellect able to solve all the questions, which it puts to itself? Must it be possible to decide on the quadrature of the circle, is there any well-defined question related to sensual things, which must have an answer and nevertheless cannot be answered? Can one show a universal and compelling proof that every cognition must be possible?

Second note: Book1, Page32, Region2
Transcription: Wenn wir auf 2 verschiedenen Wegen widersprechende Resultate finden, muss nothwendig ein Fehler enthalten sein. Was wurden wir denken mussen, wenn kein Fehler sich findet? Wir werden diesen Fall als unrealisirbar betrachten. Mussen wir in gleichen Sinne auch den Fall als unrealisirbar betrachten, dass sich etwas nicht beweisen (entscheiden durch endliche Zahl von Schlussen ⟨Schlüssen⟩ lasst)?
Translation: When we find contradictory results in two different ways, a mistake must necessarily be present. What would we have to think, if no mistake is found? We will consider this case impossible. Do we have to consider, in the same sense, the case impossible where something is never proved (decided through a finite number of deductions)?

3.8. Solvability of every mathematical problem

In the previous notes, Hilbert was not very clear about the problem of the solvability of mathematics or pure thinking. His belief in the completeness of human intellects was increasing from Kantian philosophy notes to the early formalistic notes, but he does not seem to have strong confidence in this belief.

In the note displayed in Fig. 2, Hilbert seems to have finally become confident. He clearly claims that every mathematical problem is solvable (lösbar) by a finite number of operations, and gives even a “normal form” of all mathematical problems. We will call this note *the solvability note.*

Remarkably, a problem whether ten consecutive 7 appear in the constant π, which is almost the same as Brouwer’s ”counterexample” of the law of the excluded middle, is mentioned.

Book1, Page37, Region2: solvability note

Transcription: Jedes Mathem. Problem kann auf folgende Frage zuruckgefuhrt werden: Gegeben ist ein Gesetz, vermoge dessen zu jeder vorlegten ⟨vorgelegten⟩ nur aus Null und 1 bestehenden nicht abbrechenden Reihe
(α) 0 0 1 1 0 0 ...

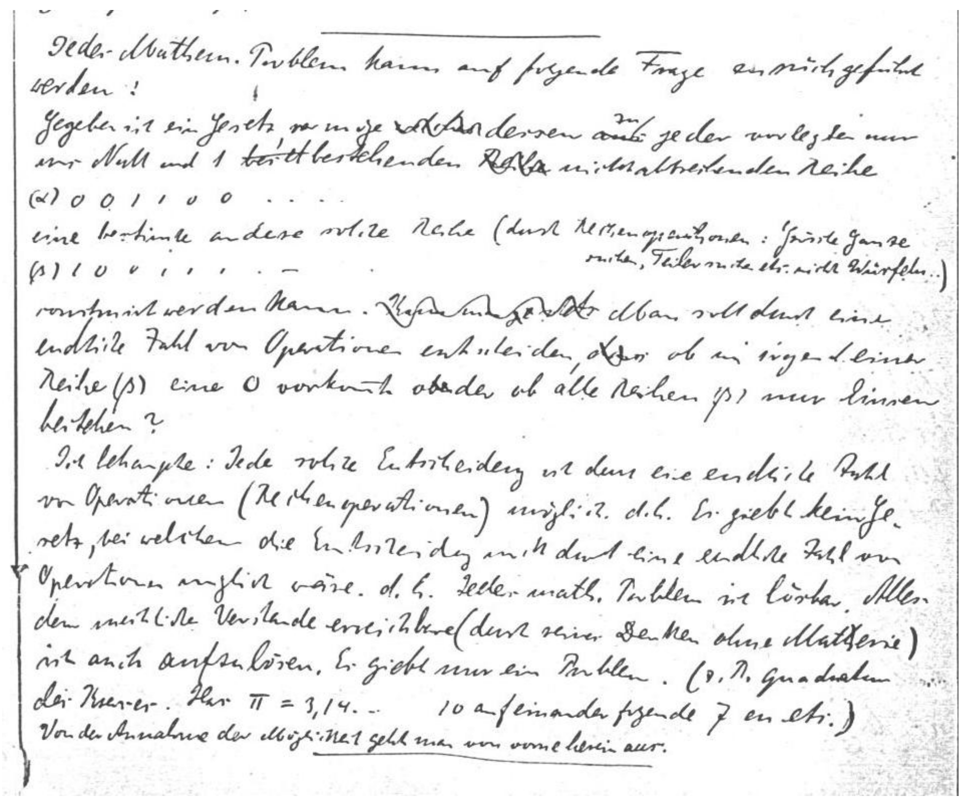

Figure 2: Solvability note

eine bestimmte andere solche Reihe (durch Rechenoperationen: Grösste Ganze suchen, Teilersuche etc. nicht Wurfeln...)
(β) 1 0 0 1 1 1....
construiert ⟨konstruiert⟩ werden kann. Man soll durch eine endliche Zahl von Operationen entscheiden ob in irgendeiner Reihe (β) eine 0 vorkommt oder ob alle Reihen (β) nur ⟨aus⟩ Einsen bestehen?

Ich behaupte: Jede solche Entscheidung ist durch eine endliche Zahl von Operationen (Rechenoperationen) moglich⟨⟨ , ⟩⟩ d.h. es giebt ⟨gibt⟩ kein Gesetz, bei welchem die Entscheidung nicht durch eine endliche Zahl von Operationen moglich ware. D.h. jedes math. Problem ist lösbar. Alles dem menschlichen Verstande erreichbare (durch reines Denken ohne Materie) ist auch aufzulosen. Es giebt ⟨gibt⟩ nur ein Problem. (z.B. Quadratur des Kreises, hat $\pi = 3,14\ldots$. 10 aufeinander folgende 7en, etc.) Von der Annahme der Möglichkeit geht man von vorne herein ⟨vornherein⟩ aus.

Translation: Every mathematical problem can be reduced to the following question: A law is given, by which to every given non-breaking off sequence consisting only of zero and 1

(α) 0 0 1 1 0 0...
a certain other such sequence (by computational operations: Grösste Ganze suchen[12], factorizations, etc. not throwing dices...)
(β) 1 0 0 1 1 1....
can be constructed. One is to decide by a finite number of operations whether zero occurs in some sequence (β) or all sequences (β) consisting of only ones.

I claim: every such decision is possible through a finite number of operations (computational operations), i.e. there is no law, with which the decision would not be possible to reach by a finite number of operations, i.e. every math problem is solvable. All that the human intellect can reach (by pure thinking without matter) is also to be dissolved. There is only one problem (e.g. quadrature of the circle, $\pi = 3.14\ldots$ has 10 successive 7's etc.) One proceeds at the outset from the assumption of the possibility.

The note is rather surely written after the middle of March 1888 and by June 30th 1891. It is, however, more likely written by the end of June 1889 or even earlier. In any case, it is written during the time when Hilbert was working on his new invariant theory based on infinite methods of [**?** 31] and is moving forward to the more constructive version by Nullstellensatz of [32, 33].

We will give the grounds of our estimations below. First, we will give grounds of the "lower bound", the middle of March 1888, of the estimations. The note of Book1, Page33, Region1 mention of his new proof of Gordan's theorem of binary case. Thus, it is after the middle of March 1888. Then, Hilbert was on a grand trip visiting many mathematicians (March 9-April 7 1888). On March 16th he wrote to Klein from Leibzig that Gordan kindly came to see him in Leibzig (see, 27 Hilbert and Klein on p.38 of [13]). As we have already noted in [**?**]), on March 21st, he again wrote to Klein from Leibzig that he could find a new short proof of Gordan's theorem for binary forms (28 Hilbert an Klein on p.39 of [**?**]).

In the following note of Book1, Page33, Region1, he mentioned his own proof of the binary case.

Book1, Page33, Region1 Mein Beweis für die Endlichkeit der Invarianten der binären Formen giebt[=gibt] mehr als der Gordansche Beweis. Denn zeigt sich zugleich, dass alle Invariante ganze ganzzahlige Funktionen einer endlichen Zahl von Invarianten sind.

[12] The phrase Grösste Ganze suchen is left untranslated, for we are not yet sure what it means. W. Sieg suggested us to interpret it the problem to find the greatest integer smaller than or equal to a given number. Nineteenth century German and Austrian elementary arithmetic books sometimes include exercises such as "Wie viel Ganze sind enthalten in $13/over5$, $31/over5$, $52/over5$, $126/over5$?" (e.g., [78], p.72). Since the following example of Rechenoperationen "factorization" is from the elementary arithmetic, it is very likely that it means this kind of fraction arithmetic exercise as Sieg suggests. However, we could not have found the exact wording "Grösste Ganze suchen (Suche[n])", although we looked through all the dozens of 19th century German and Austrian arithmetic books for elementary and secondary schools that are kept in National Diet Library in Tokyo. Since "Teilersuche" for the other example is a common word, this point remains unclear to us. Thus, we leave the problem of the reading the phrase unanswered by way of caution.

Translation: My proof of the finiteness of invariants for binary forms gives more than Gordan's proof, because it also shows that all invariants are polynomials with integer coefficients of finite numbers of invariants.

It would be very certain that Page33, Region1 was written after the middle of March 1888. Hilbert meanwhile provided yet new proof for the general case by the finite basis theorem. He submitted three papers to report the results that he obtained to Göttinger Nachrichten [30], including number theoretic considerations such as the one of Page33, Region1. As we have mentioned in 3.6, the note of Page33, Region3 and Page34, Region1 is also likely written when he was working on these three papers. The first of the three papers was submitted on October 3rd 1888 (33 Hilbert an Klein on p.43 of [**?**]). Thus it is more likely that the two notes mentioned are written in the summer of 1888. Since this "sharper" estimation is not quite informative for our purpose, we use the more conservative estimation "after the middle of March, 1888".

A rather sure "upper bound" of the time for Book1, Page37, Region2 is June 30th 1891. On that day, Hilbert submitted the first of the three preliminary papers for the famous 1893 Mathemnatische Annalen paper [33] again to Goettinger Nachrichten [32]. This time, the principal means was Nullstellensatz, which Hilbert regarded as a generalization of Max Noether's theorem.

Several notes on the theorem have been found in Book1. For example, the note on Page41, Region3 consists of "Veralgemeinung der Nötherian Fundamentalsatz auf Formen mit mehr Reihen von Variablen" (the generalization of the Nötherian theorem with more variables). This is likely a research plan, or, rather less likely, a note to record the finding of the proof. Thus, we can infer that the note on page 41 and so the note on page 37 were written before June 30th 1891, the submission day of the first paper of Nachrichten papers [32].

More circumstantial but much more natural upper bound is "the end 1888 or 1890". It is the time when the finite basis theorem was published first time. The grounds of the estimation are "Ehrensache note" of the Book1, Page53, Region2 and Book1, Page54, Region1, and "Gordan note" of Page76, Region3 and Page77, that will be discussed in 3.9 and 3.11, respectively.

Although it is not explicitly mentioned, it is clear that the work mentioned in the Ehrensache note is the 1890 Mathematische Annalen paper or its preliminary papers. Since the Gordan note is likely written, during the affair about 1890 Mathematische Annalen paper in the Feburary and March of 1890 (c.f. 2.1, 3.11), the Ehrensache note would be about the first Nachrichten paper. Thus, the Ehrensache paper would be written in 1888-89, when the first paper was published and the Berlin mathematicians saw it. It would be in the end of 1888 or early 1889, since the submission date was October 3rd 1888 and it is in the 1888 volume. Thus, so is the solvability note. Anyway, it is at the time that Hilbert was working on his revolutionary non-constructive theory of algebraic invariants and was going to aware of the new direction by Nullstellsatz.

Note that Hilbert wrote about essentailly the same "counterexample" which his adversary L. E. J. Brouwer would introduce about 35 years later. Brouwer introduced some "counterexamples" to the law of excluded middle, which are now called *Brouwerian counterexamples* (cf. [68, 69]). Brouwer showed that if such a counterexample to the law of excluded middle holds, then a mathematically unsolved problem at the time must be solved then. Namely, he reduced "invalidity" of the law of excluded middle to contemporary open problems. A typical problem of such an open problem was "if there is a sequence 0123456789 in the decimal expansion of π". This example is so famous that some people regard this example as *the* Brouwerian counterexample.

This particular Brouwerian counterexample was introduced in 1923 [**?**]. However, if 0123456789 is replaced with 7777777777, Hilbert's example of a problem of the difficulty with the solvability princile is identical to Brouwerian counterexample (see p. 270, [4], see also 10.4 in [69], [70]). Hilbert had already sensed with this example the "difficulty" of maintaining his solvability principle. Nonetheless he determined to believe in it and start from it. Many years later, Brouwer refuted the validity of Hilbert's solvability claim (in 1900) and the law of the excluded middle altogether by using the same kind of counterexamples (c.f. [4, 5, 69]) without knowing the young Hilbert's thoughts.

Brouwer's objections to the law of excluded in 1920's, perhaps, reminded Hilbert of his thoughts in his youth. Brouwer wrote about his intuitionistic philosophy and a criticism of Hilbert's solvability principle in 1900 Paris talk already in his doctor thesis. In the thesis, he admitted the validity of the law of excluded middle yet. ??? Example ??? Soon later, he started to doubt the validity of the law of classical logic. However, his biographer van Dalen (Stigt?) pointed out that these activities on the foundations of mathematics would not reach Hilbert easily, as they are written in Dutch and published in Dutch journals. It is likely that Hilbert realized Brouwer's activity on the foundations of mathematics around 1920, when Weyl began to work on the foundations.

In the next section, we will relate this remarkable note to Hilbert's research on the theory of algebraic invariant theory, especially, the finite basis theorem by analyzing his later lecture notes and papers, and also by applying a logical similarity between the principle claimed in the note and the theorem.

3.9. Hostility to Berlin mathematicians

The note examined in this subsection is not about foundations. However, it is related to the acceptance of the finite basis theorem and it will be also important to understanding Hilbert's attitudes to Kronecker, and so it is examined here.

Hilbert public attack against Kronecker's ideas on the foundations of mathematics is now very famous [**? ?**]. However, Hilbert's public attack against Kronecker was made only nine years after Kronecker's death of 1891. In his 1900 Paris talk, he did not explicitly mention Kronecker's name, although it was apparent that his foundational philosophy is against Kronecker's. In his 1905 Heidelberg talk, he even mentioned Kronecker's name as a dogmatist.

The first irrational number note shows that Hilbert was against Kronecker's philosophy already in the late 1880's. He became even hostile in the era of Grundlagenstreit [**?**]. It is widely thought that the source of Hilbert's hostility is due to the difference of their opinions on the methodology of mathematics and the constructivities in mathematics.

Evidently, a source of his very strong hostility will be the *counter-revolution* by Weyl and Brouwer[13] to his non-constructive revolution of mathematics by the use of set theoretical methods such as finite basis theorem and the early axiomatics. The first irrational number note could be regarded as evidence for the conventional belief.

Nonetheless, it was not only foundational issues which made Hilbert hostile to Kronecker. It is well-known that, in the introduction of Zahlbericht [**?**], Hilbert expressed his dislike to Kummer-Kronecker's style of number theory by heavy formula manipulations (see [52], ???). The difference of the styles between Hilbert and Kronecker is not only foundational or philosophical but also methodological. The following note shows that Hilbert's hostility was caused, or at least strongly boosted, by comments on his finite basis theorem by Kronecker and other Berlin mathematicians.

Book1, Page53, Region2 and Book1, Page54, Region1

Transcription: Von verschiedenen Quellen (von Minkowski durch Hensel und von Eberhard durch Gutzmer und andere) habe ich gehört, es herrsche in Berlin die Ansicht, "als sei in meiner Arbeiten nicht zu finden, was nicht schon im Kroneckers Untersuchen stände" (Hensel) und "meine Untersuchungen seien nur einer Uebertragung der Kroneckerschen Ideen in die Invariantentheorie (Kronecker selbst soll dies gesagt haben, wie Gutzmer dabei gewesen ist) Ein flüchtiger Blick in meine Arbeit zeigt das Gegenteil: Sogar der übrigens noch gar nicht ⟨⟨ einmal ⟩⟩ von Kronecker herrührende Begriff eines 'Moduls' ist wesentlich modifirt (duruch die Forderung der Homogenität. Alle weiteren Begriffe und überhaupt alle Methoden und Resultate sind mein. Eigenthum[Eigentum] (insbesondere auch der erste Fundamentalsatz) und ich lasse mir davon auch nicht ein Tüpfelchen nehmen von niemand, wer es auch sei. Vielmehr betrachte ich als selbstverstaendliche Ehrensache, dass, nachdem ich dies gesagt, jeder anstaendige Mensch zur Beseitigung dieses offenbaren Irrthums[Irrtums] beitragt

Translation: From different sources (from Minkowski, an opinion by Hensel, and, from Eberhard, by Gutzmer and others), I have heard the opinions "in my work, as if it is not found what had not already existed in Kronecker's investigations" (Hensel) and "my research was only a transfer of Kronecker's ideas into the invariant theory (Kronecker himself is reported to have said it, as Gutzmer was present there). A glance of my work shows the opposite: even the concept of 'Modul', which is incidentally not due to Kronecker at all, is considerably modified (through the requirement of homogeneity). All further concepts and actually all methods and results are my properties (in particular also the first fundamental theorem) and I let no one take away a bit from my work, whosoever else it might be. Furthermore, I regard it a self-evident matter of honour that, every respectable man contributes to the eliminations of these evident mistakes, as I have declared above.

[13]It is an even ironical that Weyl called Brouwer as "the revolution" [].

The hostile tone against Berlin mathematicians is remarkably strong. The work discussed in the note will be Hilbert's first solution of the Gordan problem and "the first theorem" mentioned will be the finite basis theorem. The finite basis theorem was the first theorem in the both of the Göttinger Nachrichten reports [30] of 1888- 1889 and 1890 Annalen paper [31].

The concept of 'Modul' is normally regarded to be due to Kronecker [**?**], but Hilbert denied this. On the other hand, Hilbert was pretty proud of his finite basis theorem. Interestingly, Berlin mathematicians' opinions on Hilbert's constructive invariant theory are not "incorrect" as in the case of Gordan's criticism, but "not new". Did the Berlin mathematicians including Kronecker accept the non-constructive argument of the finite basis theorem or did they simply overlook it? A plausible answer to this question would be that Berlin mathematicians had not then seen the proof. As we will point out in 3.11, even Gordan was positive to Hilbert's work on the invariant theory, when it was published in Göttinger Nachrichten without proofs. Perhaps, the Berlin mathematicians in the note above had not read the proof either. These considerations suggest that the note is likely written around 1888-1889. We will call the note the *Ehrensache note*.

3.10. Noscemus

It is well known that Hilbert regarded his solvability "axiom" of his 1900 Paris lecture as a counterargument to P. Du-Bois Reymond's agnosticism, which were famous in 19th century by a motto "Ignoramus et ignorabimus" (we do not know and we will not know).

It seems that Book1, Page72, Region3 is the first occasion when Hilbert related Du-Bois Reymond's agonticism to his solvability principle. This note will be called *noscemus note.*[14]

Book1, Page72, Region3

Transcription: Die Mathematik ist die einzige Wissenschaft, bei welcher wir nicht ignorabimus⟨, ⟩ sondern im Gegenteil im ⟨⟨ weiteresten ⟩⟩ ⟨weitesten⟩ Umfang in Bezug auf alle und noch so schwierigen Probleme sagen mussen: noscemus.
Translation: Mathematics is the only science where we are not "ignorabimus", rather, on the contrary, with the furthest extent, concerning all and even very difficult problems, we must say "noscemus".

ignorabimus: We shall not know. (Latin)

noscemus: We shall know. (Latin)

[14]This note had been reported by R. Thiele in [65].

3.11. Gordan and excluding middle?

In section 2, we illustrated the affair on 1890 Mathematische Annalen paper involving Hilbert, Gordan and Klein. The contents of the note consisting Book1, Page76, Region3 and Book1, Page77, Region1 remind us of it.

Book1, Page76, Region3 and Book1, Page77, Region1

Transcription: Zu Gordan sagen: Wenn Sie kommen, so weiss ich entweder, es ist Prof. Gordan, oder es ist nicht Prof. Gordan. Sie aber wurden sagen: oder drittens. Die kommende Person ist Keins von Beiden.
Von meinen Invariantentheorischen Satzen folgendes sagen: 1) Die Symbolische Rechnung und *erei ⟨Rechnerei⟩ in der Invariantentheorie stets an meinen Satzen in de⟨⟨ n ⟩⟩selben Verhaltniss ⟨demselben Verhaltnis⟩ wie die Jakobische Rechnerei in der Theorie der elliptischen Funtionen zu der nachfolg. ⟨nachfolgenden⟩ Theorie von Weierstrass oder von Riemann⟨,⟩ wo durch ⟨this word is not necessary⟩? die Funktion durch die Nullstelle bestimmt wird. 2) Die Symbolik hat⟨,⟩ so kann man mit ⟨⟨ ein ⟩⟩iger Ubertreibung sagen⟨,⟩ der Invariantentheorie geschadet, da meine Satze sonst vielleicht ⟨⟨ schon ⟩⟩ f⟨⟨ ru ⟩⟩her entdeckt waren, wie die Erfindung des Luftballons dem Fortschritt in den Bestrebungen eine Flugmaschine zu erfinden geschadet hat.
Translation: Say to Gordan: If you come, then I know either, it is Professor Gordan, or it is not Professor Gordan. You would however say: or thirdly. The coming person is none of both.
On my Invarianten theory theorems, ⟨I⟩ say the following:
1) the symbolic calculation and repeated calculations in the invariant theory ⟨are⟩ always in my theorems in the same proportion that Jacobi's calculation in the theory of the elliptical function ⟨are⟩ in the succeeding theory due to Weierstrass and Riemann, where the function is determined by root⟨s⟩.
2) one can say with some exaggeration that symbolism has damaged the invariant theory, as the invention of the balloon damaged the progress of the efforts of invention of airplanes, since otherwise my theorems would have been discovered earlier instead.

The first part of the note is rather strange, but the second part is very likely on Gordan's "theology" criticism. Gordan's letter to Klein suggesting rejection of Hilbert's revolutionary paper is dated February 24th 1890 and Hilbert's objection to the criticism was sent to Klein on March 3rd 1890 (see [13]). If we suppose that the note was written on the occasion of the affair, notes in the pages around Page76 were likely written in the early 1890s.

Since Hilbert published the finite basis theorem without proof in Göttinger Nachrichten 1888, one may think that there was a conflict between Hilbert and Gordan at the time. However, Gordan's letter to Hilbert dated 26th December 1888, which we reproduced in B, suggests that it is not the case. After the contents and the date, it is very likely Gordan's reply to the report sent to him. Although he expressed his reservations on the formulation of the finite basis theorem by saying that it was too general, he also wrote that he did not doubt the validity of the theorem. He repeatedly congratulated Hilbert on the great achievement. Gordan would not know the non-constructivity of Hilbert's proof of the finite basis theorem, perhaps by the time when Klein asked him to review the Annalen paper with the non-constructive proof of the theorem.

3.12. Desk and blackboard: mathematics as a theory of things

Hilbert's student Otto Blumenthal reported in his Lebensgeschichte for Hilbert that Hilbert had said "Man muss jederzeit an Stelle von 'Punkte, Geraden, Ebenen' 'Tische, Stuhle, Bierseidel' sagen konnen" (one must always be able to say 'tables, chairs, beer mugs' instead of 'points, line, plane'), when he was discussing with other mathematicians in a waiting room of Berlin station. They would be on the way back to Königsberg from Halle, where they had attended the annual meeting of Deutschen Mathematiker-Vereingungof 1891.

In the meeting, Hermann Wiener, a mathematician at Halle university talked about geometry and it is said that Wiener's talk stimulated Hilbert's idea of his axiomatics. This story was first reported by Hilbert's pupil Blumenthal in his Lebensgeschichte for Hilbert [3] and later popularized by Reid's biography [61]. However, any historical materials by Hilbert himself on this famous story had not been known. In the following note consisting Book1, Page72, Region3 and Book1, Page73, Region1, Hilbert wrote about "mathematics on systems of tables and blackboards, etc (Tisch, Tafel, etc.)". It will be worth mentioning that this note directly follows the noscemus note of 3.10. We will call this note *system-mathematics note*.

Book1, Page72, Region3 and Book1, Page73, Region1

Transcription: Mehrere Dinge zusammen in einem Begriff gefasst, geben ein System⟨,⟩ z.B. Tisch, Tafel, etc... In der Mathematik betrachten wir Systeme von Zahlen oder von Funktionen. Dieselben brauchen nicht in abzahlbare Menge zu sein. Das System ist vielmehr gegeben und bekannt, wenn man ein Gesetz kennt, vermoege dessen von einer vorgelegten Zahl oder Funktion entschieden werden kann, ob dieselbe zu dem System gehoert oder nicht. Beispiele. Die Systeme koennen Eigenschaften haben z.B. invariante Operationen z.B. Alle Funktionen gehoren ⟨⟨ w ⟩⟩ieder zum System, die man erhaelt, wenn man a) + (z.B. ositiven Losungen von linearen diophantischen Gl.⟨Gleichungen⟩) b) +- c) + und * (z.B. System aller definiter Formen) d) +- * (Integritaetsbereich) e) +- * : (Rationalitaetsbereich). Endliche - unendliche Basis − Systeme von Systemen.

Translation: Several things combined into a concept, gives a system, for example: table, blackboard (or dinner table) etc. In mathematics, we examine systems of numbers or of functions. They need not be in countable sets. The system is rather given and known, if a law is known, with which a given number or function can be decided, whether it belongs to the system or not. Examples. The system can have the properties, e.g. invariant operations, e.g., ⟨values of⟩ all functions, which one gets, belong again to the system, when one ⟨applies⟩ a) + (e.g. positive solutions of Diophantine equations) b) +- c) + und * (e.g. systems of all definite forms) d) +-* (Integral domain) e) +-*: (Rational domain). Finite- Infinite bases ___ System of systems.

Rational domain: an old name for ”field”

Interestingly, his examples of “mathematics of tables and blackboard” were not geometrical but algebraic. It is surely written before 1892 November, when Hilbert started to write the second notebook.

3.13. Another normal form: a link to axiomatics?

In the previous subsection, we saw a primitive form of axiomatics in the system mathematics note. However, the axioms of algebraic systems in modern mathematical terminologies were then called “Eigenschaften” (properties) instead. In the next two subsections, we will see the probably first occasions when he thought about axiomatics as a means of founding mathematics and physics. In the solvability note and the other early notes that we examined, Hilbert thought about the foundations of mathematics philosophically and mathematically. How are they related to his axiomatics? Perhaps, the note examined below is a link to axiomatics from the “normal form” of mathematical propositions in the solvability note.

Book2, Page14, Region2
Transcription: Jeder Satz in der Mathematik kann positio gefasst und als Existenzsatz von folgender Gesatalt ausgesprochen werden: Zu einem System von Zahlen von der und der Eigenschaft lässt sich sin System von Zahlen von der und der Eigenschaft zuordnen. Das erzte System repräsentirt das gegebene Constante, das letztene System reprasentirt das Gesuchte, Veränderliche, Beliebige. Die Eigenschaften drücken sich wieder durch solche Zuordungen aus, und so steigt der ganze Bau auf.
Translation: Every mathematical proposition can be affirmatively reformulated and be expressed as an existence theorem of the following form: a system of numbers lets its numbers and property assigned to the numbers and property of another system of numbers. The first system represents the given constant system and the second system represents the demanded, variable and aribitrary system. The properties again squeeze themselves by the assignments and so the whole structure rises up.

In this note, Hilbert gives a new normal form of mathematical propositions. It is remarkably different from the one which he described in the solvability note. In the modern terminologies of algebra and model theory, Hilbert’s normal form looks to constitute an existence theorem of a morphism from a structure to another structure. The first “constante” structure would represent the “subjects”, say a, b of a proposition $P(a, b)$. The second “gesuchte, veränderliche, beliebige” system would represent the ”predicate” P. If D is a structure in the modern mathematical sense, it is easy to express a proposition of the form $\forall a, b \in D.P(a, b)$ in the form that Hilbert described.

Note that Hilbert considers a “System” of “Zahlen” and “Eigenschaften” on the numbers as a framework of expressing mathematical propositions. Hilbert’s early axiomatics or ur-axiomatics in the system mathematics note had this pattern. Actually, he used the same terminologies “System” and “Eigenschaften”.

The difference is that Hilbert considered not only systems of numbers but also systems of "functions" such as forms (homogeneous polynomials with many variables). Since Hilbert mentioned the repetitions of assignments, it would have been possible to reduce a normal form of systems of forms onto normal forms of systems of numbers of the coefficients of the forms. Of course, as we do not know what Hilbert had in mind, our considerations must be speculative.

If our speculation is acceptable, the systems in this note and the systems in the system mathematics note would be the same or very similar concepts. Then, the note might be considered to link the normal form of mathematical problems in the solvability note to the axiomatics through the system mathematics note.

3.14. Axiomatization of Physics

Recently, several authors have pointed out the importance of Hilbert's physics studies [7, 54]. Corry argued that many features of Hilbert's early axiomatics had been borrowed from the foundational thoughts of some physicists, notably from Heinrich Hertz. In a sense, the foundations of physics precedes the foundations of geometry in Hilbert's thoughts.

Corry's proclamation was based on similarities between Hertz's idea and Hilbert's idea, and a quotation of Hertz's name in Hilbert's lecture notes ([**?**], p.762). The note of Book2, Page16, Region2 will be a textual piece of evidence supporting Corry's viewpoint. It does not mention Hertz's name, however, it strongly suggests that Hilbert's original motivation was not axiomatization of geometry but of physics. See 4.5 for further discussions. We will call the note *the physics note.*

Book2, Page16, Region2

Transcription: Suche die Axiome der Mechanik (und später auch die der Optik, Electricit"atstheorie etc.) genau so aufzustellen wie es mit den Axiom, der Geometrie geschehn ist. Dabei suche die den Axiomen zu Grunde liegenden Experimente genau auf und beschreibe die ganze Erscheinungen fülle. Beweise auch warum ein anderes Sysmtem von Axiomen die Erscheinungen schlechter beschreibt. Führe letzteres zuerst für die Geometrie genau aus.

Translation: Search for the axioms of mechanics (and later also the ones of optics, electricity theory etc.) to place them exactly with the axioms, as geometry happens to do so. At the same time, search for experiments which the grounding axioms rest upon and describe the phenomenon fully. Prove why other axiom systems of the phenomenon are described worse. Do it first for geometry.

The note is written after November 1892, since Book2 dated November 12th 1892. The note on "existence=consistency" of the next subsection is estimated to be written in or before 1893/4. Thus, it is quite likely that the physics note is written in the end of 1892 or in 1893. It might be written in 1894, but it is not plausible.

3.15. Existence=consistency

Hilbert's thesis of "existence=consistency" is very famous. The history of the idea that appeared in his lecture notes on geometry in the 1890's was carefully

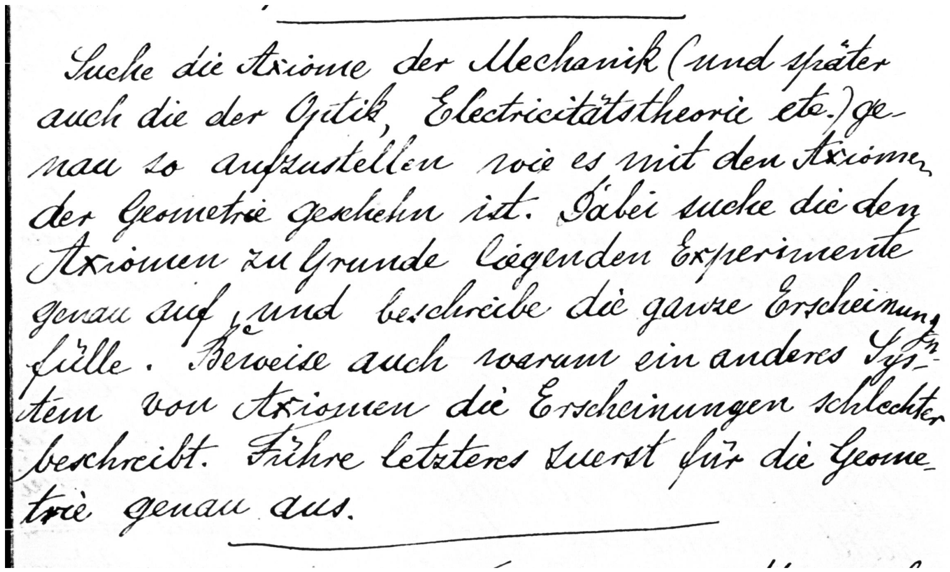
Suche die Axiome der Mechanik (und später
auch die der Optik, Electricitätstheorie etc.) ge-
nau so aufzustellen wie es mit den Axiomen
der Geometrie geschehn ist. Dabei suche die den
Axiomen zu Grunde liegenden Experimente
genau auf, und beschreibe die ganze Erscheinungs-
fülle. Beweise auch warum ein anderes Sys-
tem von Axiomen die Erscheinungen schlechter
beschreibt. Führe letzteres zuerst für die Geome-
trie genau aus.

Figure 3: Axiomatization of Physics

analyzed in the first volume of the Hilbert edition by Hallet and Majer [15]. See the editor's footnote 173, pp.282-283 of the volume and Hilbert's own writing annotated by it, the second paragraph of p.104 of the 1898/1899 lecture notes "Grundlagen der Euklidian Geometrie", Cod. Ms. D. Hilbert 551. We reproduce here the paragraph from the volume:

> Existiren heisst die den Begriff definirenden Merkmale (Axiome) widersprechen sich nicht, d. h. es ist nicht möglich aus allen ⟨Axiomen⟩ mit Ausnahme eines durch rein logische Schlusse einen Satz zu beweisen, welcher dem letzten Axiom widerspricht.

The phrase "⟨Axiomen⟩" is an editorial addition by the editors of

It was pointed out that Hilbert's idea of "existence=consistency" appeared already in the lecture notes of 1894 summer semester lecture course on geometry "Grundlagen der Geometrie". However, in the 1894 lecture notes, Hilbert did not say "Existenz" but said "Möglichkeit"(quoted from pp.119-120 of [15]):

> Thöricht ist es, wie z.B. Lotze thut, die Möglichkeit der nicht Euklidischen Geometrie von vorneherein zu verwerfen. Die M'oglichkeit derselben, d. h. die innere Widerspruchslosigkeit derselben, kan vielmehr strenge gezeigt werden.

The definition of "existence" rather than "possibility" was alluded in 1896 Holiday course as "Dass die *Eukilidishe Geometrie existirt*, folgt aus der *Existenz* der analytische Geometrie". The phrase was quoted from p.167 of [15]. See also the editor's footnote 27 of the same volume.

The two consecutive notes displayed below, Book2, Page17, Region2- Book2, Page18, Region1 and Book2, Page18, Region2, correspond to Hilbert's statements on "existence=consistency" thesis illustrated by Majer and Hallet. In the former note, Hilbert talked about "the possibility of Euclidian geometry", and give the definition of existence in the latter note.

Remarkably, the sentence in the second note in which Hilbert defined existence by consistency explicitly is reproduced verbatim in the lecture notes in 1898/9. This shows that Hilbert intentionally related his early foundational thought to his study of the foundations of geometry in 1898/9.

Book2, Page17, Region2 and Book2, Page18, Region1
Transcription: Die Moeglichkeit der Euklidischen Geometrie - und damit auch die dier nicht Euklidische Geometrie, d.h. die Widerspruchslosigkeit der Axiome in sich, kann man leicht analytisch zeigen, in dem man jeden Punkt durch 4 homogene Coordinaten bestimmt. Suche jedoch diesen Beweis rein geometrische nachzubilden. vergl. dazu meinen Gedankengang eines "ueber Euklidische Geometrie" zu lesenden Collegs am Schluss.
Translation: The possibilities of the euclidian geometry - and therefore also those of the non-euclidian geometry - which means the absence of contradiction of the axioms themselves, can be shown easily in an analytic way by defining every point with 4 coordinates. But try to reproduce this proof with pure geometry. Compare this to my train of thoughts ⟨in "about euklidean geomety" for my current lecture at the conclusion.⟩

Book2, Page18, Region2
Transcription: Existiren ⟨Existieren⟩ heisst die den Begriff definirenden ⟨definierenden⟩ Merkmale (Axiome) widersprechen sich nicht, d. h. es ist nicht möglich aus allen mit Ausnahme eines durch rein logische Schlusse einen Satz zu beweisen, welcher dem letzten Axiom widerspricht. Man braucht jedoch, zumal im gewohnlichen Leben existieren ⟨"existieren"⟩ auch gleich für die Beschreibung der Dinge besonders forderlich sein, z. B. Gott existirt ⟨existiert⟩ nicht, d.h. das Axiom Gott ist, ist zur Beschreibung vieler Erscheinungen uberflussig
Translation: To exist means that the conditions defining the concept do not mutually contradict, i.e. from all ⟨axioms⟩ with an exception ⟨an exceptional axiom⟩, it is not possible to derive a proposition, which contradicts to the last ⟨exceptional⟩ axiom. One needs however, particularly in ordinary life that "exist" is also equally useful for the description of the things particularly, e.g. God does not exist, i.e. the axiom "God exists", is superfluous to the description in many cases.

Hilbert would have written these notes to memorize his first ideas of the axiomatic ontology. We will call these notes *the first and second ontology notes.* Unfortunately, any good estimations of the writing time of the two ontology notes are not available. The first notebook ends in November 1892, and Hilbert is known to have left the area of algebraic invariants and moved to algebraic number theory in the fall of 1892. He submitted a paper on algebraic number theory already in the fall of 1893. The note right after the second note presented above is on determinants of number fields which were later considered in 1897 Zahlbericht.

After Majer's introduction of the 1894 lecture notes of [15], the lecture was originally planned to be deliverer in the 1893 summer semester, but was cancelled because of too few registrations. Hilbert is likely to have written the lecture notes already in 1893. However, we cannot exclude the possibility that he wrote them in 1894 when he actually taught the lecture course. Thus it would be safer to guess that the notes were written in 1893 or 1894. We have not extensively studied the time of these notes yet. More careful analysis of the notes in the second notebook would give more precise estimation.

4. Hilbert's early thoughts on the foundations of mathematics

In the previous section, we presented some notes from the first and second notebooks together with their philological and textual analysis. We avoided speculative interpretations as far as we could. When we read texts with some speculative assumptions, the assumptions were explicitly indicated.[15] Although even such a restricted approach already suggests some important conclusions, such as that Hilbert already has his solvability conviction very early in his career, it inevitably gives less information than interpretative reading. In this section, we will draw more conclusions by interpretative readings.

4.1. Kronecker's influences

The first irrational number note would be the first note on the foundations in the three notebooks. However, the tenth problem note written before the first irrational number note could have a foundational meaning, although there is no certain evidence for it. Anyway, these two early notes, perhaps from 1888 or a little bit older, show Kronecker's influence on Hilbert's foundational thoughts.

Kronecker declared that mathematics should be reduced to natural numbers, e.g. [50, 51]. The real meaning of his "reduction of mathematics to the natural numbers" is not a reduction to the natural number arithmetic but a reduction to his theory of Modul systems[9] Correct reference?????. His theory of Modul systems (or divisor theory) was a theory of Moduls, which are algebras of many-variable integer-coefficient polynomials modulo, given polynomials of the same sort. It was originally invented to rationalize Kummer's ideal numbers, and was a rival to Dedekind's ideal theory. Kronecker called the algebraic theory of Modul systems "general arithemetic".

Hilbert's famous solution of Gordan problem is well-known to make use of Kronecker's general arithemetic. They say that that Hilbert solved the long standing open problem of algebra named after Gordan by his unexpected application of Kronecker's arithemetical theory. Hilbert was quite familiar with Kronecker's theory of general arithmetic.

[15] A notable example is the assumption on the notes on Study's letter.

As we noted in 2.1), Hilbert visited Kronecker in Berlin in his trip from May to April 1888, Kronecker told the irrational number $\sqrt{5}$ by his general arithmetic and Hilbert wrote down it in the memorandum for the trip [**?**]. This would mean that Hilbert did not know about the number until then.

Kronecker wrote that every mathematical concept should be decidable in a footnote of a paper on the general arithmetic, and criticized Dedekind's ideal theory on this basis (see ??? [10]). Later in the 1920s, Hilbert criticized this opinion. It is not so difficult to reduce the problem of existence of solutions of equations in a Modul system to a problem of existence of solutions of diophantine equations.

Thus, if the tenth problem could be solved positively against reality, Kronecker's condition on the decidability of concept is alway automatically met as far as mathematical concepts are defined by equations of Module systems. Thus, if one accepts Kronecker's view of pure mathematics (e.g. 20 D in [10], [9], [50, 51]), then a very wide range of problems of pure mathematics can be reduced to a problem of diophantine arithmetic. Thus a positive solution of the tenth problem implies the solvability of many problems of pure mathematics. Did Hilbert take the idea of the solvability of every mathematical problem after conceiving of the possible positive solution of the tenth problem? At the moment, we do not have enough evidences for a plausible answer.

4.2. From Kantian philosophy to the solvability problem

Influence of Kant's philosophy on Hilbert is often mentioned. The first irrational number note of 3.3 and Kant notes of 3.6 show that Hilbert in very early time followed Kant's original viewpoint of mathematics based on the intuition of time and space. However, he also started to build his own standpoint soon.

The first irrational number note is the first note on irrational number in Hilbert's notebooks. Kronecker prohibited the general notion of irrational numbers, since it could not be realized by his general arithmetic. Hilbert strongly argued against Kronecker's prohibition in the note.

The reason of his opposition is "the irrational numbers should be accepted if the rational numbers are accepted." The rational number 1/2 is regarded as an "idealization" of the center of a line segment. Hilbert saw transgression (Überschreiten) from pure intuition to pure formal thinking already in the concept of the rational number 1/2. Then, one should accept other idealization such as $\sqrt{2}$, if he accepts the idealization of the rational numbers such as 1/2. Hilbert also maintains that such a transition of idealization opens a wide path for mathematics. These are Hilbert's reasons to accept irrational numbers.

In the first irrational number note, Hilbert does not mention Kant's name explicitly, but he uses the notions from Kantian philosophy such as "Anschauung", "anschauen" and "vorstellen". Integers are considered to be based on the intuition of space and time as in Kantian philosophy. Even the structure of the counterargument to Kronecker reminds us Kant's counterargument to Hume.

Kant argued against Hume's objection to the possibility of metaphysics by changing the system of metaphysics from the system of the laws of the real and absolute world to the laws of our own intellect. Kant made metaphysics possible by shifting its position. If we call "objective" positions higher and "subjective" positions lower, we may say that Kant lowered the position of metaphysics to save it. Hilbert took a similar strategy to defend the notion of irrational numbers from Kronecker's skepticism. He made the irrational numbers possible by lowering the position of the rational numbers which Kronecker accepted.

The inserted phrase "vielleicht streng bloss von 1 bis 5 höchste[höchstens] 7" suggests that he even tried to lower the absoluteness of positive integers, for which Kronecker said to be given by God. The inserted phrase is rather unclear and it is difficult to draw a definite conclusion. However, it is worth pointing out that, in his 1904 Heidelberg talk [39], Hilbert called Kronecker a "dogmatist" who accepts integers as the starting point of mathematics without examination. He claimed that even the existence of positive integers had to be proven by a consistency proof of their axioms. If Hilbert's formal systems are regarded as mathematical formalizations of "rein formal Gedachten", then the idea of the first irrational number note conforms with the philosophy of his formalistic programs in 1900's and 1920's. It seems that Hilbert repeatedly use this pattern of thinking as counterattacks to agnosticism in his academic life. See discussions at the end of 4.4.

In the Kantian philosophy notes, the influence of Kantian philosophy is even more apparent. In the first note, Hilbert poses an axiom of human intellects (Verstande). He answers by himself with a reservation, "Vielleicht dass jedes Problem lösbar ist?". The phrase is written with small letters and gets shifted slightly upwards. See Fig. 1. It might be added later, or might be written with hesitation then. In the second note, Hilbert explicitly claims that the notion of integers is based on time and space. He explicitly mentions Kant by name as the source of these thoughts.

In the third note, Hilbert talked about the "origins" of the materials of the sciences. He wrote "allen anderen Wissenschafte". It is not clear which "Wissenschaft" he had in his mind and wrote about "anderen Wissenschafte" to it. "Reines Denken" and "reine Philosophie" are mentioned in the note. On the other hand, the second note is on the notion of integers. In the early formalistic notes, he continued philosophical considerations on the boundary of human intellects, and, in the first of these notes, the problem of the quadrature of the circle is mentioned in the context of the ability of human intellects generally. In the solvability note, he maintained the solvability of all mathematical problems and *also* all the problems which the human intellect can reach by pure thinking without matter (see the transcription and translation of 3.8).

Based upon this evidence, we suppose that the "Wissenschaft" about which Hilbert was writing in the third Kantian philosophy note must be *mathematics and pure reason*. It seems that mathematics was taken as an instance of pure reason and the other pure reasons were supposed to have the same nature as mathematics. Probably, mathematics is a "generic" pure reason or the "representative" of pure reasons for Hilbert. The notion of solvability of mathematics in the solvability note that we will examine in the following subsection must be the answer to the metaphysical questions in the early formalistic notes and

Kantian philosophy notes.

Hilbert's discussion on the origins of mathematics and other "Wissenschafte" in the third note reminds us his later discussions on Emil du Bois-Reymond's "Ignorabimus". A renowned German physiologist Emil du Bos-Reymond presented a famous work of skepticism in his 1880 talk [8]. His skepticism is often referred by the Latin sentence "Ignorabimus" (we do not know and we shall not know) closing the talk or simply "Ignorabimus". He listed some problems which never will be solved by science or philosophy. Among them were the origins of motion and simple sensations (????).

Hilbert asserted that, in mathematics and pure thinking (or pure philosophy), the origin of materials can be known. Interestingly, he attributed this idea to Kant. Then he wrote that *Kant did not prove it.* It is likely that Hilbert wished to prove a philosophical proposition. It is not clear which philosophical question he wished to prove in Kant notes.

However, it is made clear in the first of the early formalistic notes. Although it is stated in the form of question, Hilbert started to talk about *a proof* of the completeness of human intellects: "Kann man einen allgemeinen zwingenden Beweis fuhren, dass jede Erkenntniss $\langle\langle$ möglich $\rangle\rangle$ sein muss?". In the Paris talk, the question is stated in a vague way. One could regard it a mere incentive for mathematicians but not a serious proclamation. Since we know the metamathematical results derived by Gödel, Turing and others, it would be more natural to think for our purposes that one of the greatest mathematicians in history claimed that we human beings can give a proof of our mathematical omnipotence.

The philosophical notes quoted above show that Hilbert was serious about the validity of the principle and its proof. In the early formalistic notes, Hilbert raised a question which reminds us of Gödel's incompleteness theorem: the possible existence of a question which is true but cannot be answered by human beings by a finite numbers of deductions. In the second of the early formalistic notes, Hilbert tried to answer it positively by a philosophical argument. However, the answer was not convincing enough. He then gave another "answer" in the solvability note.

4.3. Solvability as the origin of Hilbert's foundational thoughts

The solvability note, the Kantian philosophy notes, and the early formalistic notes suggest that Hilbert seriously thought about decision methods of mathematics, by which every mathematical problem could be solved only by computations. However, Hilbert did not clearly state his conviction in solvability in public papers and talks. For example, in the 1900 Paris lecture where he expressed his "principle" for the first time, he called the principle an "axiom". He pointed out that some difficult problems of mathematics such as the squaring of the circle are negatively solved by proofs of the impossibility of positive solutions, and then continued (p.1102, [10]):

> It is probably this important fact along with other philosophical reasons that gives rise to the conviction (which every mathematician

> shares, but which no one has as yet supported by a proof) that every definite mathematical problem must necessarily be susceptible of an exact settlement, either in the form of an actual answer to the question asked, or by the proof of the impossibility of its solution and therewith the necessary failure of all attempts.

He clearly stated his conviction of the solvability of mathematics in the sense of the solvability note. The phrase "but which no one has as yet supported by a proof" reminds us his arguments in the notes on Kant and solvability.He even called the principle as an axiom:

> Is this axiom of the solvability of every problem a peculiarity characteristic of mathematical thought alone, or is it possibly a general law inherent in the nature of the mind, that all questions which it asks must be answerable?

If he had stopped here, many would have had impression that Hilbert took the investigation of solvability of mathematical problems as a serious scientific or at least philosophical matter. However, he continued to state probably the most famous phrase of the talk:

> This conviction of the solvability of every mathematical problem is a powerful incentive to the worker. We hear within us the perpetual call: There is the problem. Seek its solution. You can find it by pure reason, for in mathematics there is no ignorabimus. (English translation from

By calling the principle of solvability an *incentive* for mathematicians, this phrase changes the impression of the preceding discussions. It changes the solvability of mathematical problems from a principle to a sermon of industriousness to mathematicians. In the last part of the phrase, he mentioned the Ignorabimus dispute on du Bois Reymond's famous philosophical skepticism of science, which was widely discussed in the middle 19th century intellectual scene in Germany.

It will be fair to say that the phrase that we cited last made the overall tone of the discussion on the solvability rethorical. Most people would think that the discussion on solvability is not a serious scientific conjecture but a slogan. L. E. J. Brouwer was an exception. He identified Hilbert's solvability of Paris talk with the principle with the principle of excluded middle in logic and attacked both of them [5]. However, one of Brouwer's biographers van Dalen, who is himself a notable logician, expressed his puzzlement on Brouwer's serious objection to the solvability principle. To van Dalen, Brouwer's act looked unnecessarily serious, since the solvability claim in Paris was "one of those slogans that belong rather to popular lectures than to science proper" (p.586, vol.2 [69]). Most modern mathematicians would agree with van Dalen.

Hilbert might have intentionally avoided proclaiming such a controversial statement openly. However, in a semi-closed circumstance, Hilbert expressed very clearly his conviction as a goal of his scientific research into the foundations of mathematics. Furthermore, he then mentioned that it was the original motivation of his foundational studies. In the 1905 summer semester, Hilbert gave a lecture on the logical foundations of mathematical thoughts [40]. In the lecture notes, Hilbert established a decision method of propositional logic in a somehow vague way, and then commented on it as follows (English translation is from 2.2.1 of [79]):

> So it turns out that for every theorem there are only finitely many possibilities of proof, and thus we have solved, in the most primitive case at hand, the old problem that it must be possible to achieve any correct result by a *finite proof*. This problem was the original starting point of all my investigations in our field, and the solution to this problem in the most general case⟨⟨ , ⟩⟩ the proof that there can be no “ignorabimus” in mathematics, has to remain the ultimate goal.[16]

He clearly stated that the ultimate goal of his investigation of mathematical logic is to prove the correctness of his conviction of “kein Ignorabimus” found in the 1900 Paris lecture. It is notable that he described it as the motivation of his research into mathematical logic. The statements quoted above might look odd, if “kein Ingnorabimus” in the 1900 Paris lecture is interpreted as nothing but a motto or an incentive for mathematicians. However, if we interpret it as a refined form of the solvability principle of his youth and the 1900 “kein Ingnorabimus” as a statement of that conviction, they look to be quite natural. The 1905 research into completeness would be Hilbert’s early attempt to mathematically proving his solvability principle.

There is an interesting question regarding the phrase quoted above. Hilbert called the problem of solvability *the old problem* “das alte Problem”. The “axiom” of his 1900 lecture was then only 5 year old. He never published the principle of solvability in his youth. Thus it is not likely “das alte Problem”. Prof. Michio Kobayashi pointed out that “das alte Problem” would be related to Leibnizian theory of two kinds of truth and proofs. Leibniz distinguished necessary truth and contingent truth. The necessarily true propositions can be proved by finite steps of logical rules such as the principles of identity and contradiction. However, the contingently true propositions can be proved only by infinite proofs, which could be conducted only by God (e.g. [57]). Leibniz identified most mathematical truth as necessary, but the truth of propositions on infinitesimals as infinite and so contingent.

There is no historical evidence for Kobayashi’s theory. However, not only

[16]So ergibt sich hier, das für jeden Satz nur endlich viele Beweismöglichkeiten existieren, und wir haben damit in dem vorliegenden primitivsten Falle das alte Problem gelöst, das jedes richtige Resultat sich durch einen *endlichen Beweis* erzielen lassen mus. Dies Problem war eigentlich der Ausgangspunkt aller meiner Untersuchungen auf unserem Gebiete und die Erledigung dieses Problemes im allerallgemeinsten Falle der Beweis, das es in der Mathematik kein Ignorabimus geben kann, mus auch das letzte Ziel bleiben.

Kobayashi but also several philosophers and logicians have pointed out similarity of Hilbert's ideas in the foundational studies and Leibnizian philosophy of truth. A notable example is Ian Hacking, who related Leibnizian infinite proofs to Hilbertian infinitary proofs with ω-rule in [16, 17], and emphasized influences of Leibnizian philosophy on Hilbert's foundational works. Hilbert's infinitary proofs which appeared in his 1931 paper [44] are widely believed to be a counterargument to Gödel's incompleteness theorem. However, Bernays reported that Hilbert was working on the idea of infinite proofs before Gödel's theorem [**?**]. Bernay's report would be not unnatural if Hilbert had associated his solvability conviction to Leibniz's theory of truth.

4.4. Why was the solvability principle acceptable?

Historical evidences presented in the previous subsections show that Hilbert not only believed in the existence of an universal "computational" method by which one can solve any mathematical problem, but also that the pursuit of such a universal method was the original motivation of his research into the foundations of mathematics. Such a universal computational method, however, would trivialize mathematics, as Hilbert's pupil John von Neumann once wrote (pp. 11-12 [71]):[17]

> And the undecidability[18] is even the Conditio sine qua non in the sense that it had a meaning to make mathematics work by the today's heuristic methods. On the day when the undecidability ends, mathematics in today's sense also ends to exist; in such a situation an absolutely mechanical instruction will come in, and by the help of the instruction every man can decide every given statement provable or not.

The solvability will kill mathematics by leaving nothing to do for mathematicians except mechanical computations. This would be nightmarish for most mathematicians.[19] Hilbert is known as one of pioneers of the non-computational methods in the twentieth century modern mathematics and disliked computational methods in mathematics. Why did Hilbert believe the existence of an

[17] Und die Unentscheidbarkeit ist sogar die Condition sine qua non dafür, daßes überhaupt eine Sinn habe, mit den heutigen heuristischen Methoden Mathematik zu treiben. An dem Tage, an dem die Unentscheidbarkeit aufhörte, würde auch die Mathematik im heutigen Sinne aufhören zu existieren; an ihre Stelle würde eine absolute mechanische Vorschrift treten, mit deren Hilfe jederman von jeder gegebenen Aussage enscheiden könnte, ob die bewiesen werden kann oder nicht.

[18] The decidability here is not the decidability of validity of formulas, but decidability of their provability in given formal systems.

[19] We should note that computations in mathematics have been turned out not to be trivial in any sense by the advancement of experimental or computational mathematics. Even with the helps of super computers, many algorithms for mathematics are too difficult to execute. Researches on innovative algorithms and better ways of executing algorithms have been proved very creative intellectual activities, although the style of such researches is rather different from the one of the twentieth century modern mathematics.

universal computational method to solve every mathematical problem? In the rest of this subsection, we will give a probable answer to this question. First, we will examine how Hilbert disliked mathematics by computation. Then, we will argue that he could, however, believe in the existence of a mechanical method solving every mathematical problem without trivializing mathematics, since he knew that some mechanical methods for solving mathematical problems are simply infeasible. If a universal decision method is not feasible at all by its computational complexity, then mathematicians do not need to worry about the nightmare that von Neumann illustrated. We will also point out that Hilbert knew the difference of feasible algorithms and infeasible algorithms even before his solvability principle through his research into the algebraic invariants.

First, we will illustrate why Hilbert disliked computational mathematics. In the introduction of his seminal report on algebraic number theory, Hilbert wrote with respect to his theory of Kummer fields that he tried to avoid Kummer's large computational apparatus (den groß en rechnerischen Apparat von Kummer zu vermeiden) so that Riemann's principle "complete proofs not by computations but purely with ideas" was realized for the case. He also praised Cantor's theory of transfinite numbers in the introduction. Laugwitz let this episode symbolize the turning point of mathematical methodology from computations with formulas to pure thinking with ideas (4.3, [52]).

Before championing computational method to non-computational method in 1897, Hilbert had already used non-computational methods in his theory of algebraic invariants. Nontheless, he had not given any explicit criticisms on computational methods. However, in the notebooks, Hilbert had often written about his opposition to computational methods earlier and much more clearly. [20] The note of Book1, Page34, Region2 reads " formulas can only describe thoughts, cannot replace or save them: machine manufacturing —– handicraft."[21] It is written in between the early formalistic notes and the solvability note. It is written before the Gordan note, which also shows his dislike for computations. A later but even clearer example is Book2, Page103, Region2 "Anything, which is an object of thinking, is therefore an object of mathematics. Mathematics is not the art of computation, but the art of non-computation."[22] Compared to these direct and strong comments, the one in the introduction of Zahlbericht are very polite and mild.

Probably, the most striking example illustrating Hilbert's thoughts on mathematics done solely by computations will be the discussions on "decidability" of mathematical problems in the talk "Axiomatisches Denken" 1917 [41]. In the talk, he mentioned five epistemological problems on mathematics which should be challenged in his foundational study program even after solving consistency (c.f. [**?**] for detailed discussions). Among them, he chooses the problem

[20]Thiele[65] has reported some of such notes.

[21]Die Formel kann das Denken (den Gedanken) nur beschreiben, nicht ersetzen, order ersparen: Maschinenfabrikat —- Handarbeit

[22]Alles was Gegenstand des Denkens ist, ist daher Gegenstand der Mathematik. Die Mathematik ist nicht die Kunst des Rechnens, sondern die Kunst des Nichtrechnens.

of “das Problem der Entschedbarkeit einer mathematischen Frage durch eine endliche Anzahl von Operationen” as the most famous and discussed problem (die bekkanteste und die am häufigsten diskutierte) among them and explain in details by using two examples. The first example is his two proofs of Gordan’s problem. He pointed out that the problem of “Entscheidbarkeit” was solved on this special example by his second proof enabling the computation of solutions, although his first proof could not. Namely, “Entschedbarkeit” was not meant to be the modern sense of decidability, but meant the constructivization of non-constructive proofs. The modern terminology “Entscheidbarkeit” as decidability was introduced years later by Hilbert’s student Behmann. See [53]. When it is necessary to distinguish these two concepts of Entscheidbarkeit, let us call the “Entscheidbarkeit” in 1917 paper *1917 Entscheidbarkeit* and call the modern one *modern Entscheidbarkeit*.

In the second example of the problem of 1917 Entscheidbarkeit, Hilbert considers a problem from the theory of algebraic surface. The problem is to determine the maximum number of separate sheets it takes to make up the algebraic surface of the fourth order. We will below quote from the English translation (pp.1114-1114, 24 C [10]). He wrote “the first step towards an answer to this question is the proof that the number of sheets of a curved surface must be finite.” Hilbert pointed out this can be done very easily by function theory, although the function-theoretic proof does not give the upper bound for the number of surface-sheets at all.

He then pointed out that the problem can be formulated in a calculation problem in 34-dimensional space: “it is accordingly possible to establish, by a long and wearying but finite calculation, whether we have a surface of fourth order with $n \leq 12$ sheets or not.”[23] It is a decision method of a particular but important mathematical problem. He then pointed out that the problem had been reduced to a problem of the level of difficulty of determining the $10^{(10^{10})}$th numeral in the decimal expansion for π, which was a task clearly solvable, but remained unsolved. Computing $10^{(10^{10})}$th numeral in the decimal expansion for π was not only impossible but also unimaginable in Hilbert’s time.[24]

Then he further pointed out that a German mathematician Karl Rohn had taken an entirely different algebraic-geometric investigation to show 11 sheets are not possible. Hilbert emphasized that the full solution (die völlige Lösung)) of the problem is given only by Rohn’s method.

In these passages of [41], Hilbert showed that a difficult and important problem of mathematics could be reduced to a huge computational problem. How-

[23]es is daher möglich, durch eine endliche, wenn auch sehr müsame und langwierige Rechnung, festzustellen, ob eine Fläche 4. Ordnung mit $n \leq 12$ Mänteln vorhanden is oder nicht.

[24]The computation is still impossible now even by contemporary advanced algorithms for π, such as BBP-algorithm, and supercomputers. David Bailey, an inventor of BBP-algorithm, kindly answered Hayashi’s question on the problem. He wrote Hayashi that the computation is “fundamentally beyond the reach of what we can do with known algorithms such as BBP, even if we had a computer the size of the known universe. Perhaps sometime in the future someone will discover a much more efficient algorithm, ... although I doubt it.”

ever, it could be so huge that the computation itself is an impossible task to do in practice. Then, the problem could be solved by entirely different solution without computing the reduced computational problem, and *that this is the real way to do mathematics*.

Hilbert's arguments illustrate that a computational decision method solving a mathematical problem could be impractical due to its large amount of computations, and giving a mathematically deep proof without computations is the real way of mathematics. Now, it would be clear how Hilbert could accept the solvability principle without fears of trivializing mathematics.

There is some other evidence which support this interpretation. In the first edition of the textbook of mathematical logic by Hilbert and Ackermann[48], chapter 3, section 11, Hilbert and Ackermann discussed the decision problem (modern Entscheidungsproblem) of Funtionenkalkül, the higher-order ramified predicate logic in the modern terminology. They wrote the solution of the problem has fundamental importance.

Nonetheless, after the emphasis of the importance, they also wrote as follows:[25].):

> We wish make it clear that a procedure were given by the solution of the decision problem so that, at least in principle, any unprovability had to be established with it, even if, probably, the intricateness of the procedure could make the practical execution illusory.

This phrase was deleted from the later editions after Gödel-Turing works. The many contents of book had already appeared in Hilbert's 1922/23 winter semester lecture note Cod. Ms. D. Hilbert 567 "Logischen Grundlagen der Mathematik". The phrase had appeared verbatim in the lecture notes.

Even a universal decision method for every mathematical problem, the mathematics might not be trivialized at all due to the huge computational complexity of the method. Note that we actually have a similar situation in reality. It is not difficult to build a piece of software called a *theorem prover for the first order formal system of ZF set theory.* If the memory of a computer is infinite as a Turing machine, such a theorem prover can automatically find all theorems of ZF set theory, thus almost all mathematical theorems in the history of mathematics, e.g. proofs of Fermant's last theorem and Poincare conjecture. This is a well-known fact which is even taught in elementary courses of logic for computer science. However, in reality, such a theorem prover cannot automatically prove even very elementary theorems, since the complexity of the computations is astronomical. The Coditio sine qua non of the conventional mathematics is not non-existence of theorem provers but their infeasibility.[26]

[25]Wir wollen uns klar machen, daßmit Lösung des Entscheidungsproblems ein Verfahren gegeben wäre, durch das jene Unbeweisbarkeit sich wenigstens grundsšatzlich festsetellen lasssn müsste, wenn auch vielleicht die Umständlichkeit des Verfahrens die praktische Durchführung illusorisch machen könne. (p.74, [48]

[26]In that sense, von Neumann was wrong, It will be worth to remark that, more than a

Recall that Hilbert wrote a reservation "assuming that there were no practical difficulties", when he wrote on the solvability of mathematics in the first note of the early formalistic notes. Perhaps, he then believed that any method solving every mathematical problem would be infeasible, and thus thoughts on the problem of solvability of mathematics was meaningful only when the problem of huge quantity of computations was ignored.

Namely, he likely knew about the difference between "computability in principle" and "computability in practice" already in late 1880's. The notion of computational complexity and computationally difficult problems such as NP-problems are now among the common sense of mathematicians and computer scientists, but they were developed only in the late 20th century. How did a young Königsberg mathematician of the late 19th century get to know such difference?

Of course, many people know after their own experiences that there are different kinds of computational problems: some are easy to calculate and others are not. In the time when a large part of mathematics was done by formula computations rather than modern style abstract thinking, many mathematicians would have experienced such difference through their own researches. Hilbert was an example of such mathematicians. As we have wrote in 2.1, he had known a typical example of infeasible algorithms in his study of algebraic invariant theory.

In the winter semester of 1886/1887, the year before the breakthrough of the finite basis theorem, Hilbert gave a lecture on the theory of algebraic invariants. The lecture was given in the traditional style by formula computations including Gordan's theorem for the binary invariants. However, he gave a very interesting comments on the theorem and its algorithm to compute full invariant systems, complete form systems in the note below:

Cod. Ms. D. Hilbert 521, pp.193-194

Transcription: Gordan zeigt auch dabei gleichzeitig, wie mann zu einem solch wollständige Formensystem gelangen kann. Doch ist die Methode so complicirt ⟨kompliziert⟩, dass man sie nur in den ersten Füllen bis $n = 8$ hat anwenden können. Wir haben das vollstä⟨⟨ ndige ⟩⟩ Formensystem ⟨⟨ für ⟩⟩ $n = 1, 2, 3, 4$ aufgestellt und für $n = 5$, $n = 6$ noch in §15 noch die Invariante angegeben. Wir begnüge⟨⟨ n ⟩⟩ uns mit den gemachten Angaben, da die gewaltig angeschwollene Litteratur; doch nur weitere Beispiele für die <u>einfönigste und trokenste</u> symbolische Rechnung liefert. Das <u>wichtig und fundamentale</u> ist allein jener Satz von der Englichkeit der Formensystemene.

Translation: At the same time, Gordan also showed how one can reach to such a complete form system. However, the method is so complicated that one has been able to make use of it only in the first cases to $n = 8$. We have given

decade later, von Neumann wrote the other way around. In the famous book "Theory of games and economic behavior" [73], he and Morgenstern proved the existence of the winning strategy of chess game. Then, they commented that their winning strategy is not practical due to its monstrous size and it makes the chess game a *good* game.

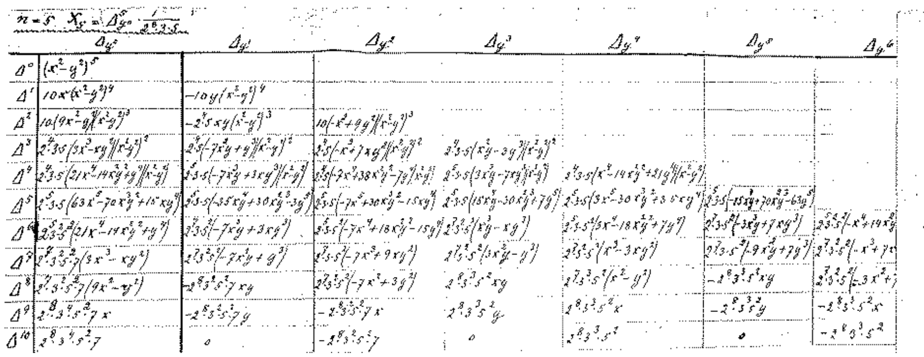

Figure 4: Hilbert's formula tables

the complete form system ⟨⟨ for ⟩⟩ $n = 1, 2, 3, 4$ and for $n = 5$, $n = 6$, yet in §15, invariants as well. We are content with the worked out results, as the huge swollen literature adds however only further examples of the most monotonous and dullest symbolic computation. What important and fundamental is soley that theorem on the finiteness of form systems.

This passage shows that Hilbert knew already in 1887 of a problem computable in principle but not computable in practice. The difficulty of the computation of Gordan's algorithm would be well-know by Hilbert, as Gordan published such computations for a few specific cases. Nonetheless, it seems that most algebraists are still attacking the problem by complicated computational methods. Before the turning point in 1887 lecture, even Hilbert was not an exception of human formula computers and compiled formula tables for his dissertation as in Fig.4 (from Cod. Ms. D. Hilbertxo ???).[27]

Hilbert changed the way to attack the Gordan Problem in 1888 and he succeeded in solving it by the a non-constructive method. The solvability note was written around the time when he found the finite basis theorem. Thus, it is very likely that he had an idea on the difference between computation in principle and computation in practice at that time. In "??", it will be pointed out that Hilbert discussed the computations in mathematics from time to time in his academic career and he often related the discussion with his two solutions of the Gordan Problem.

After the work on the two solutions of the Gordan Problem, Hilbert even once envisaged a theory of "computational complexity". The following note from Book2 is likely written in the end of 1892 or so:

Book2, Page0a, Region3 and Book2, Page0b, Region1

[27] It was usual for contemporary mathematicians to compile such a formula table. For example, folded sheets of long formulas or their tables are inserted in some volumes of Mathematische Annalen.

Transcription: Stelle auf eine Theorie des Rechnens d.h. ergrüde worin die Vortheile ⟨Vorteile⟩ bekannter arithmetischer Formeln und Sätze (Summation arithmetischer Reihen, Congruenz⟨Kongruenz⟩-formeln, Stirlingsche Formel, vergl. Schlömilch Compendium Bd 2 S.264.) zum Ausrechnen bestimmter Aufgaben beruhen und wie man diese Vortheile rechnerisch abschätzen kann. Vielleicht indem man gewisse elementare Rechnungsoperationen (addiren ⟨addieren⟩, multipliciren ⟨multiplizieren⟩, potenziren ⟨potenzieren⟩) als Einheiten einführt und die Zahl der anzuwendenden Einheiten auf verschiednen Wegen, die man beim Rechnen einschlägt, bestimmt. Wie viel Rechnungeinheiten z.B. sind nötig, um die so und so vielte Stelle der Decimalbruch-Entwicklung von π zu bestimmen? Wie berechnet man n! am schnellsten? Vielleicht die ersten Ziffern aus Stirlings Formel, und die letzen durch Congruenz⟨Kongruenz⟩-Rechnung.

Translation: Set up a theory of calculations i.e. find out on what the advantages of well-known arithmetic formulas and theorems (summation of arithmetic series, congruence-formulas, Stirling's formula, c.f. Schlömilch's compendium vol.2 p.264) to calculate certain tasks are based and how one can estimate those advantages by way of calculation. Perhaps ⟨⟨ that would be done ⟩⟩ by introducing certain elementary calculation-operations (add, multiply, power) as units and determining the number of units to be used in different ways, by which one takes in the calculation. How many calculation-units, for example, are necessary in order to decide an arbitrary place of decimal-development of π? How does one calculate n! in the fastest way? Perhaps the first numbers by Stirling's formula, and the last ⟨⟨ numbers ⟩⟩ through congruence-calculation.

Before we close this subsection, we will point out that Hilbert's strategy for accepting solvability is morphologically similar to the one of his defence of irrational numbers and the Kantian strategy of the defence of metaphysics on which we discussed in 4.2. Hilbert dismissed the widely accepted reliability of rational numbers by regarding them as the ratio on line segments and pointing out the practical impossibility of identifying them absolutely. By lowering the reliability of the rational numbers in this way, he dismissed Kronecker's boundary between the rational numbers and irrational numbers, and maintained that the irrational numbers should be accepted provided the rational numbers are accepted.

For the case of solvability, he dismissed the omnipotence of the universal mechanical solution method of mathematics by maintaining its practical infeasibility based on the infeasible universal mechanical solution methods of particular problems such as Gordan's algorithm and the algebraic solution method for Rohn's algebraic geometrical problem. By regarding the universal solution method of mathematics to be on the same level of these existing methods, the universal solution method is no wonder to exist. Namely, both arguments are based on the same strategy of discussion: countersigning to objections against an expectedly absolute existence by eliminating or lowering the expectations on the absoluteness of the existence. It might be worth pointing out that Hilbert's rescue of Riemann's Dirichlet principle from Weierstrass' counterexample and even the "existence=consistency" ontology follow similar patterns of argument.

4.5. Early notes on axiomatics

Hilbert's foundational thoughts have two characteristics: the solvability of mathematical problems "kein Ignorabimus" and the axiomatics with the ontological principle "consistency = existence". These ideas appeared publicly in 1900 Paris talk for the first time and were kept through his academic life since then. We have examined the former with historical evidence from his notebooks and drawn new conclusions. We will examine the origin of the latter in the same vein in this subsection.

Although the early formalistic notes have some signs of axiomatics, they are still very vague. The system-mathematics notes of 3.12, the physics note of 3.14 and the notes on consistency and existence in 3.15 should be regarded as Hilbert's earliest writing on his axiomatics. Let us examine these three notes in chronological order below.

First, we examine the system-mathematics note. Although we do not have any certain sharp estimation of the upper bound of the time of the system-mathematics note, we can draw an interesting conclusion on it provided a probable assumption on the note Book1, Page80, Region2. The note is about Eduard Study's letter and reads as follows "Answer to Study's letter dated 27/6.91. Right through these shallow way to rave as the one which was introduced by Clebsch's school, the deeper mathematical creativity will be paralyzed" [28]

It is a reaction to Eduard Study's letter dated June 27 1891 kept in the file Cod. Ms. D. Hilbert 396. The sender's address is Marburg, Germany. If there was no trouble, the letter would be delivered a few days after June 27 1891. [29] Since the content of the letter is simply an expression of contempt to Study, it would not take long for Hilbert to write it. Thus it is probable that the notes on Study's letter and so the system-mathematics note were written before the Halle meeting of DMV, which took place in September of the same year. This conclusion is coherent with the fact that the system-mathematics note does not show the characteristics of Hilbert's axiomatics of the later years, as we will discuss below. Note that he did not use the word "axiom" in the note. Hilbert would have got the idea of mathematics of systems of things such as tables and blackboards before Wiener's talk on axiomatics of geometry in Halle meeting, and after the talk, Hilbert related his own idea of system-mathematics to axiomatics of geometry, and told his colleagues about geometry of tables, chairs, beer mugs in the Berlin station on the way back to Königsberg.

Corry points out in pp. 19-20 of [7](also in [6]) that Hilbert gave a discussion of axiomatics in the introduction of his 1893 invariant theory paper[33]. Hilbert

[28]Antwort auf Study Brief vom 27/6.91. Grade durch diese flaeche Art zu phantasiren⟨⟨ phantasieren, ⟩⟩ wie sie durch die Schule von Clebsch eingefuhrt ⟨⟨ wurde ⟩⟩, wird die tiefere mathematische Erfindungsksaft gelähmt.

[29]The letter No. 53 in [13] from Hilbert to Klein is dated Feb. 15, 1890. Klein's reply is the letter No. 54 in [13] date Feb. 18, 1890. Klein wrote that he received Hilbert's letter "yestderday evening". Thus it was delivered in a few days from Königsberg, East Prussia to Göttingen, Niedersachsen. The two cities are about 900 km far way, and Marburg is about 115 km further from Göttingen.

gave a list of properties of the systems of invariants to which his theory is applicable and slightly argued logical dependence and independence between the properties. Hilbert wrote that "Die zusammengestellten 5 Sätze regen die Fragen an, welche der aufgezählten Eigenschaften sich gegenseitig bedingen und welche getrennt von einander für ein Funktionensystem möglich ist." (The listed 5 propositions raise the question of which listed properties may mutually imply and which may be disconnected each other for a function system.). The paper was submitted in September of 1892 about one year after the Halle meeting. However, this argument had already appeared in the first note of the three early reports [32] of the 1893 paper. The submission date of the first note is June 30 1891 and the paper appeared in the July 1891 volume of the journal. In the first report, he listed 6 properties (instead of 5 in 1893 paper) and wrote almost verbatim "Es bietet sich die Aufgabe, zu untersuchen, welche der aufgezählten Eigenschaften sich gegenseitig bedingen und welche getrennt von einander für ein Funktionensystem möglich ist." Note that the first report was submitted 3 days later than the time when Study sent his letter to Hilbert (quoted in 4.5). If our assumption about the note on Study's letter is correct, at the time, Hilbert would have had conceived of an idea of "mathematics as a theory of systems of things" and had presented it in his early report on the results of 1893 paper as well as the system-mathematics note.

Hilbert would have conceived of some kind of semi-axiomatic viewpoint on mathematics already in the early summer of 1891, as he presented it in the system-mathematics note. The typical models of system-mathematics were algebraic systems as invariant systems as described in his 1891 report [32] on the theory of algebraic invariants. It is likely that he had the idea of dependence/independence analysis of such systems as he mentioned it in the report. And all of these events took place before he listened to Wiener in the autumn of the same year. Thus, Hilbert would have had germs of the main ideas of his famous axiomatics in his mind, when he listened to Wiener. Nonetheless, it is worth pointing out the earlier semi-axiomatic ideas in the system-mathematics note and the invariant theory papers are rather different form his later axiomatics of "Grundlagen der Geometrie" those that followed. To distinguish these two, let us call the former *system-mathematics* and the latter *Hilbert axiomatics*. Although Hilbert is one of founders of the modern axiomatics, his axiomatics is rather different from the modern axiomatic method in the sense of Bourbaki structuralism, which we will call "Bourbaki axiomatics" below. In the doctrinal paper on their structuralism [2], Bourbaki called the early axiomatics by Dedekind, Peano and Hilbert as *mistakes* (see p.230, [2]), and tried to remedy their mistakes by their new axiomatics. Interestingly, the earlier idea of system-mathematics is closer to Bourbaki axiomatics than the later idea of Hilbert axiomatics. Let us illustrate these below.

Among the differences of the two styles of axiomatics that Bourbaki discussed in [2], we now need to consider the following two for our discussions:

1. Hilbert's axiom system is "univalent". Bourbaki's axiom system is "plural".
2. Hilbert axiomatics is ontological. Bourbaki axiomatics is non-ontological.

"Univalent" is the *categoricity* of axiom systems in the present mathematical terminology. Let us quote Bourbaki's own writing on it from [2]:

> The first axiomatic treatment and those which caused the greatest stir (those of arithmetic by Dedekind and Peano, those of Euclidean geometry by Hilbert) dealt with univalent theories, i.e.,theories which are entirely determined by their complete system of axioms; for this reason they could not be applied to any theory except the one from which they had been extracted (quite contrary to what we have seen, for instance, for the theory of groups).

Modern axiomatics consider mainly the *plural* axiom systems such as the theory of groups and rings, which many essentially different mathematical structures satisfy. On the contrary, Hilbert's main target of his axiomatics is the theory of *the* real numbers. Hilbert axiomatics was first published in a short article [36] of 1899. In the article, he compared with his axiomatics with *the genetic methods* of Dedekind and Cantor, i.e., the methods of deifying the real numbers by the cuts or the fundamental sequences of rationals. In the article, he publicly declared the ontological principle "consistency of an axiom system implies the existence of the mathematical objects *uniquely* determined by the axiom system" for the first time.[30] Note that univalence of an axiom system is inevitable for the ontological principle.

In the sense, the idea of the system-mathematics note is closer to Bourbaki's modern axiomatics rather than ontological Hilbert axiomatics. Note that Hilbert used the words "Gestz" and "Eigenschaft" rather than "Axiom" in the system-mathematics note. It is not likely that Hilbert related the idea of system-mathematics to the axiomatics, when he wrote the system-mathematics note. The system-mathematics note and writings in algebraic invariant theory papers suggest that Hilbert had already conceived of the idea of study of the dependency between axioms and mathematics as a theory of abstract objects such as chairs and tables before he listened to Wiener and started his axiomatic studies in geometry.

In the physics note, Hilbert wrote about a research plan of axiomatization of mechanics and some other areas of physics. He planned first to axiomatize the physical theory and then to analyze the axiom system thus installed. He then wrote: "do it first for geometry."

In the second ontological note appearing two pages after the physics note, he gave his famous definition of the "existence" of mathematical objects, i.e., a mathematical object exists when its axioms are consistent. Interestingly, the sentence of this definition is identical to the sentence of the definition of "existence" in 1898/9 winter semester lecture on geometry. It is known that Hilbert had discussed the "possibility of Euclidian geometry" (die Möglichkeit der Euklidischen Geometrie) by "consistency" in the lecture notes of 1893/4 winter semester (see pp.119-120, [15]). It is also known that Hilbert wrote about the

[30] Of course, the objects are determined uniquely only up to isomorphism.

"existence" of Euclidian geometry (die Euklidische Geometrie existirt) by the existence of analytic geometry in the lecture notes for the holiday courses of 1896 and 1898 (see pp.167, [15]). The first ontological note right before the second ontological note is about determining the possibility of Euclidean geometry by proving the consistency of the axioms of geometry by means of the model of analytic geometry. The notes in the lecture notes will be reflections of the notes from Book1.

The task that Hilbert raised in the physics note is finding an axiom system of mechanics, showing its accordance with the natural phenomenon through experiment, checking its ability of explaining phenomenon and finally showing its legitimacy by proving its superiority to other axiom systems. These clearly represent Hertz's criteria for good physical theories as images of the nature after Corry's discussion in 1.3.3 of [7].

The physics note might suggest that the axiomatics of physics rather than geometry is the original motivation of Hilbert's axiomatics. He wrote that "beweise auch warum ein anderes Sysmtem von Axiomen die erscheinungen schlechter beschreibt. führe letzteres zuerst für die Geometrie genau aus." This phrase could be read as indicating that his original motivation was the axiomatization of physics and the "metamathematical" analysis of the axiomatization, and that the axiomatic analysis of geometry might be a kind of "rehearsal" for the project for physics. Since the ontological notes appear after the physics note, the chronological developments of Hilbert's ideas on axiomatics would be axiomatics of physics, axiomatics of geometry, the ontological principle on geometry and then the ontological principle on real numbers.

The places of the notes on the thesis of "consistency = existence" 3.15 in the second notebook seem to be coherent to the dates of comments on the thesis in the lecture notes pointed out by Hallet and Majer. These are written, when he left the theory of algebraic invariants and moved to the number theory and geometry.

It should be noted that there is a historical fact which appears in contradiction to the early reception of the idea. In their 1897-8 correspondences on Cantor's antinomy[**?**], Cantor wrote to Hilbert about his idea that the "consistency set exists", and Hilbert quoted it, when he explained his idea for the first time to the public [36**?**]. If his idea is prior to Cantor's, why did he refer Cantor's later and even narrower idea? We cannot give a definite answer to this problem, since Hilbert's correspondence with Cantor have been lost. However, it might be explained in the following way. Hilbert's writings published or unpublished show his deep respect for Cantor. He then still felt Cantor superior to himself and thought it worth appealing to Cantor's authority to explain his own idea.

4.6. Algebra and foundations

In the previous subsections, we have argued that the origin of Hilbert's thoughts on the foundations of mathematics go back to the time of his research on the theory of algebraic invariants. We also pointed out that his experience of computations in the theory of algebraic invariants might provide a reason to believe the solvability principle. In this subsection, we will further argue that his algebraic studies would have deeply influenced his early foundational thoughts and, although gradually weakened, this influence would last to the end of his academic career. It is even likely that Hilbert modeled his philosophy and metamathematical theories on the foundations of mathematics on his experiences and general algebraic theory developed, e.g. finite basis theorem and syzygy theorem, in his studies of algebraic invariants.[31]

In the lecture notes of 1917/18 lecture "Prinzipien der Mathematik", Hilbert maintained that mathematical logic is an application of the formal method of algebra to the area of logic:

> B. Mathematische Logik
> Der Logik-Kalkül besteht in der Anwendung der formalen Methode der Algebra auf das Gebiet der Logik. (page 63, Mathematische Institut Göttingen)

The contents of the lecture are largely reproduced in the 1928 book "Grundzüge der Theoretischen Logik". In the introduction of the book, there is a similar but slightly different opinion "Die theoretische Logik, auch mathematische order symbolische Logik genannt, ist ein Anwendung der formalen Methode der Mathematik auf das Gebiet der Logik", and it was pointed out that the rules of propositional logic is analogous to those of algebra.

Hilbert introduced several semi-formal logical systems from the beginning of the twentieth century. The earliest system was given in a 1904 talk at the international conference of mathematicians in Heidelberg [**?**]. It has less algebraic characteristics. The system introduced in the lecture notes "Logische Prinzipen des mathematischen Denkens"(Cod. Ms. Hilbert 558a) of the next year can be considered as a system of algebraic logic, see [79]. There is a blank for Hilbert's foundational interests after the lecture course of 1917/18 mentioned above. A reason of the revival of interest seems to be the appearance of the Principia Mathematica, see [**?**]. From the 1917/18 lecture course, Hilbert's formal systems become more "modern" as in the system of Principia Mathematica, and less algebraic. For example, the notion of quantifier is introduced in the 1917/8 lecture course for the first time. It will be worth pointing out that, in the 1920's, Hilbert made his formal systems "algebraic" again by introducing the ϵ-symbol.Furthermore, his first "planed proof" of the elimination of ϵ-symbols

There is a set of documents which collectively suggest relationship between Hilbert's algebraic works and foundational thoughts. We will examine them in chronological order below.

[31]It should be noted that a few authors have already related Hilbert's algebraic works to his foundational concerns. Smorynski related the finite basis theorem to proof theory [63], although he did not give any evidence. Corry pointed out that dependency/independency between axioms is already mentioned in his 1893 paper on the theory of algebraic invariants [6, 7]. Laugwitiz argued that Hilbert's algebraic work is a symbol of a turning point in the method of mathematics in the 19 century [52].

In April 27th 1897, 17 days after completing Zahlbericht, Hilbert started an introductory lecture on the theory of algebraic invariants [35, 47]. Three copies of the lecture notes are kept in Göttingen Main Library and Mathematical Institute Library, and in Cornell University Mathematics Library. All of them were taken by Sophus Marxsan, Hilbert's student from U.S.A, and they are practically identical. The Cornell version was translated into English and published [47].

The contents of the lecture course was Hilbert's own theory of algebraic invariants. In the lecture of July 13th, he gave a discussion on the problem of mathematical existences and computations to clarify the non-computational aspects of his first solution of the Gordan problem, whose proof, in the binary case, was sketched in the last several lectures. To illustrate the non-computability of his proof, he used essentially the same example on the decimal expansion of π in the solvability note. We quote the discussion from English translation [47]:

> With each mathematical theorem, three things are to be distinguished. First, one needs to settle the basic question of whether the theorem is valid; one has to prove its existence, so to speak. Second, one can ask whether there is any way to determine how many operations are needed at the most to carry out the assertion of the theorem. Kroneclcer has particularly emphasized the question of whether one can carry it out in a finite number of steps. Third, it has to be actually carried out; this is the least interesting question. We illustrate these questions with an example. One can ask: Is there a place somewhere in the decimal expansion of $\pi = 3.14159\ldots$ at which there appear ten consecutive ones 1111111111? It is not improbable that this may be the case. If we assume one could prove this in some way, then one can ask the second question: Can one find a number N of which one knows that there are at least ten consecutive ones before the Nth decimal of $\pi = 3.14\ldots$? The number N can be much too large, as long as we can only prove the assertion for it. Third, one would then actually have to calculate the number N_1 so that N_1th up to the $(N_1 + 9)$th decimal are all ones, and so that there are no ten consecutive ones appearing earlier; or one might have to calculate the first $N_1 + 9$ decimal of π.
>
> We too faced with a similar qeustion. We have proven that a binary form of order n has a finite full invariant system. Suppose that the degree of the invariant of highest degree in the simplest full invariant system is N; then the question arise: If we are given the number n, can we calculate an upper bound N? Our proof has settled the finiteness question only in principle; there is not the slightest indication that we can actually calculate such a number N. In one consider $n = 307$, for instance, then the proof does not indicate any number of which one would know that it is bigger than N. We will have opportunity in the next section, however, to consider this question further. The third question would here amount to actually calculate the invariants which form the full invariant system. (We will not consider this question, which is pointless for base forms of higher order.)

Note that Hilbert again uses the same "Brouwerian counterexample" in the solvability note. This time, however, what Hilbert considered is not the problem to decide if the sequence 1111111111 appears in the decimal expansion of π, but the problem to prove the existence of the sequence without knowing where it appears, and then to estimate the place where 1111111111 appears. By the terminology of constructive logic, the solvability principle is the law of excluded middle and the transition from the first step to the second step in the discussion above is called *Markov's principle* (see [**? ?**]).

At the end of the quotation, Hilbert announced that he would consider the second step of the Gordan problem in the next sections. What is considered in the next sections is Nullstellensatz and its application to solving the second step of the Gordan problem.

In some papers of the proof theory in the 1920's and the 1917 article of "Axiomatisches Denken", he repeatedly used his two solutions of Gordan's problem to illustrate the difference of computational and non-computational solutions of existence theorems. The discussions quoted above show that he had been quite aware of the problem of constructively or computationally in mathematics already in 1890's with the respect to his theory of algebraic invariants. Although his algebraic theory is less discussed than the geometric work in the context of foundational research, it seems that there was a good amount of consensus and discussion on the problematic aspects of Hilbert's non-constructive solution of Gordan's problem. We have already illustrated Paul Gordan's and Franz Meyer's attitudes to the proof in 2.1. Gordan note and papers in 1920's suggest that Hilbert was even angry about the hostility to his revolutionary work. It is then natural that Hilbert would discuss the general aspects of existence and computation in mathematics when he lectured on his work.

It would be worth noting that the non-conductivity of the work seems to have been noticed by other mathematicians, only after Hilbert submitted his 1890 Annalen paper, which included the first published proof of the finite basis theorem as explained in 3.11. Thus, Hilbert's philosophical thoughts in the notes before the Gordan note would not be motivated by criticisms by others.

In the same year when he gave the lecture course on the algebraic invariants, he was taught the set theoretical antinomy by Cantor. After that his concerns regarding the foundational issues became apparent to the public by his work on the foundations of geometry, the paper *Über den Zahlbegriff*[36], the Paris lecture [37] and the Heidelberg lecture [39]. In these articles, we cannot find any evidence relating his foundational thoughts to his early algebraic works. Although his 14th problem of the Paris lecture was the extension of the solutions of the Gordan problem, it was stated as a simply algebraic problem.

Perhaps, the only evidence bridging Hilbert's foundational and algebraic works around the turn of the century is the note on "24th problem". The note consisting Book3, Page25, Region2 and Book3, Page26, Region1 was found by Rüdiger Thiele and reported in [65]. We will call the note *the 24th problem note.* Since the 24th problem note is transcribed and analyzed in details in [65], we will not reproduce it.

The theme of the note is to find criteria of the simplicity of mathematical proofs, which will be later mentioned in *Axiomatisches Denken.* Remarkably, in the note, Hilbert suggested to use his syzygy theorem developed in his study of the algebraic invariants as a model of the notion of simplicity. This is a rare case in which he explicitly related his algebraic work to a foundational problem. Thiele argued that the note would be written in 1901 by a philological argument using notes on Werner Boy's dissertation and Otto Blumenthal's habilitation (see p.9 of [65]).

As Hilbert suggested to using an algebraic theorem to modelize the notion of simplicity of proof, it seems that Hilbert had a tendency to model logic after algebra. Actually, in the 1904 lecture [39] at Heidelberg and the lecture course given in 1905 [40], logic was formulated as a kind of algebraic theory. It is only in the 1917/8 winter semester that Hilbert started to lecture on his theory of the foundations with the modern form of logic developed by Frege, Peano, and Russell, as will be pointed out later.

From the 1905 Heidelberg lecture to the 1917 "Axiomatisches Denken", Hilbert's concerns on the foundations of mathematics disappeared from the public scene. Hilbert's main activity of this period was studies on the foundations of physics and related mathematics such as the theory of integral equations (see [**?**]). However, Mancosu points out that a change in his work already began around 1914. After this year, Hilbert and his co-workers' activities on the foundations of mathematics increased (see [53]). A possible reason for this is the publication of Principia Mathematica in 1910, 1912 and 1913. Before that time, Hilbert's logic was "algebraic" so that logical formulas are equations. After that time, Hilbert and his co-workers used logical theory *a la* Principia Mathematica. It might be worth pointing out that the main figure of this activity, Behmann, started his work on the Principia already in 1914, and the next year the publication of the Principia was completed, although his work was interrupted by the First World War for a few years [53].

In 1916, Hilbert delivered his famous talk "Axiomatisches Denken" in the capital of neutral Switzerland, and the talk was published as a paper in the next year [41]. The paper is virtually the manifesto of the beginning of the period of proof theory studies, and it shows the directions that should be pursued in the research plan. Some of them were actually practiced and the others were not. Thus we may find Hilbert's *original* motivations of the later foundational studies, and we can find strong and clear relationships to his early algebraic works in them. We will examine them below.

Hilbert used his two solutions of Gordan's problem to illustrate the problem of 1917 Entscheidbarkeit. Note that the problem of the computation of $10^{(10^{10})}$th numeral in the decimal expansion for π was used to illustrate the problem of 1917 Entscheidbarkeit as we discussed in 4.4.

From the viewpoint of mathematical logic, modern Entscheidbarkeit and 1917 Entscheidbarkeit are interrelated. Although this relationship was known only after the development of the subjects in the 1920s, it is likely that Hilbert sensed it to some extent. Some evidence for this view is his writing in 1917 "Axiomatisches Denken."

In 4.4, we noted that, in the article, Hilbert listed five epistemological problems for his new axiomatics program. They are (i) modern Entshceidbarkeit of mathematical questions, (ii) checkability of mathematical investigation, (iii) criteria of the simplicity of mathematical proofs, (iv) the relationship between formalism and content, and (v) 1917 Entscheidbarkeit. After listing them, Hilbert wrote “we cannot rest content with the axiomatization of logic until all questions of this sort and their interconnections have been understood and cleared up” (⟦42⟧, p.1113, [41]).

The third problem would mean “Hilbert’s 24th problem” which we have discussed above. In the note that Thiele found, Hilbert related the problem of simplicity of mathematical proofs to his syzygy theorem introduced in his theory of algebraic invariants (see [65]). Since the fifth problem (1917 Entscheidbarkeit) was discussed in the relation to his two solutions of Gordan problem as pointed out in 2.1, at least, two of the five epistemological principles are related to his theory of algebraic invariants.

The remaining three problems would not be related only to algera but to mathematics in general. It will be worth pointing out that they might be all related to Hilbert’s early foundational thoughts. The first problem is apparently related to his solvability axiom. The meaning of the second “checkability problem” is not clear, but perhaps, it would mean the problem discussed in the second Kantian philosophy note. The fourth problem would be the completeness problem discussed not only in the early foundational notes but also in many occasions through Hilbert’s career.

Hilbert’s work on the foundations changes considerably around “Axiomatisches Denken” works. As Zach pointed out in [79], Hilbert’s

1904 lecture: logic was algebraic

epsilon symbol and epsilon theorem and the proof of epsilon theorem

5. Conclusion

What and how Hilbert achieved in his studies of the foundations of mathematics has been well-explored by Sieg, Mancosue, Corry and other authors. However, his motivation of the studies of the foundations of mathematics has not yet been studied enough.

They say that his motivation was to save mathematics from the catastrophie caused by antinomies of set theory. Someone says that Gordan’s criticism on Hilbert’s non-constructive proof of the basis theorem was a motivation [63]. However, our research presented in this paper show that Hilbert’s interest on the foundations of mathematics were very general and philosophical. even before that he have some hostile reactions from Berlin mathematicians on his work on theory of algebraic invariants.

Our research show that it goes back to the age of his algebraic studies.

Hilbert’s questions on the foundations of mathematics in the notes on Kantian philosophy presented in 3.6 show that he was pondering even the problems of consistency and completeness of mathematics. Furthermore, he had a version of the solvability of every mathematical problem in very firm form at least ten years prior to its public appearance in 1900 Paris talk. And all these took place before Gordan’s criticism on the non-constructive proof of the basis theorem.

Since the foundations of geometry is Hilbert's first published work on the foundations, some people, especially non-experts, think that the work on geometry is the origin of his foundational works.

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A. Philological remarks and conventions

David Hilbert's Nachlaßare kept in Göttingen university library (Niedersaechsische Staats- und Universitätsbibliothek Göttingen) and Mathematische Institut of Göttingen university. The collections at Mathematische Institut are lecture notes of 1910's to 1920's. The collection at the library includes numerous unpublished lecture notes, letters and other documents. The documents of the collection at the library are numbered by its owen numbering system. The three books of mathematical notes that we are studying consist the documents Cod. Ms. Hilbert 600:1-3 in the code system.

A.1. Index system

In this paper, we will call the document Cod. Ms. Hilbert 600:1 as Book1, and the other two will be called Book2 and Book3 in the same way. All three books consist of short notes. The pair of facing left and right pages is called a *spread*, e.g. fig. 2 shows the spread of pages 36 and 37 of Book1. In all of the three books, the right hand side page of each spread is numbered, e.g. the right hand side page of the spread of 2 is numbered as 37. We will denote the pages such as Page1, Page2,..., etc.

The notes are almost always separated by horizontal lines.[32] We number the rectangular regions of each page separated the lines from the top to the bottom. The top region will be numbered by 1 and the region will be named as Region1. We will number the other regions in the same way. For example, page 36 of Book1 in the fig. 2 consists 5 regions and page 37 consists of 2 regions. A note sometiems are made of plural regions from plural pages. For example, Region5 of Page36 of Book1 and Region5 of Page36 of Book1 consist of a note. We will use index such as "Book1, Page17, Region2". This refers the second region of page 17 of the first book.

A.2. Philological conventions

Hilbert's documents are sometimes unclear and/or include errors of spellings and grammar, c.f. A of 4.2 of "Introduction to the Edition" in [15]. Thus, transcriptions must sometimes be speculative. If we are not very sure with our guess, we indicate such as "Coeffi⟨⟨ ciente ⟩⟩". This means that the part surrounded by ⟨⟨ $\cdots$ ⟩⟩ is merely our guess from the context etc. and/or we are not very sure with our guess. Note that "Coefficient" is spelled "Koeffizient" in the modern standard German spelling system. However, Hilbert spells many words in different ways (c.f. Introduction of [15]). We indicate their modern and standard counterpart as "Coeffi⟨⟨ ciente ⟩⟩ ⟨Koeffiziente⟩". However, we sometime omit such annotations, when it is easy to guess the modern counterparts. We use the same notation ⟨remarks or comments⟩ to insert comments in transcribe texts. These conventions are largely borrowed from [15].

Book1 and Book3 start with page 1. Book2 starts with page 0 but the page which should be numbered page 2 is numbered as page 1. The first and second pages of Book2 will be named page 0a and 0b respectively and the next page is named as page 1.

A.3. Philological assumptions and remarks

Since the notes are not dated at all, there is no obvious evidence that regions and notes are written in the chronogical order. In fact there are regions which are ordered against to the assumption. Book2, Page95, Region5 continues to Page94, Region1, and Book2, Page97, Region4 continues to Page96, Region1.

The reason why these notes are written in the opposite way is unknown. However it is clear that they are composed to two notes after the contents and handwritings[33]. Since they are located in the consecutive spreads, it might be done intentionally by some reason. We regard them exceptions and keep the assumption that the notes are in principle older if the pages are younger. Note that this assumption does not maintain that something was inserted, corrected or deleted later.

[32] An exception is the note on Page8 of Book1. It occupies exactly the page. Probably it is the reason why there is no separation lines before and after the note.

[33] The note of Page95 is by David Hilbert and the other is by Käthe.

We end this appendix by the comments on David's and Käthe's handwritings. From Book1, Page75, Region5, notes are mainly written by Käthe Hilbert, David Hilbert's wife. In Book1, Page80, Region2, there is a reference to a letter from Eduard Study[34] dated "27/6. 91." From the contents of the note and letter, it is very likely that the note is promptly written after its receipt. Thus, Käthe wrote notes for David, before she married him.

After David's letters to F. Klein and others, it is known that Käthe as his fiancée was writing David's mathematical manuscripts even before the time of their marriage. This is not very strange, since David's and Kathe's families were quite close even before their engamement, c.f. [61].

Occasionally, David Hilbert wrote by himself even after Käthe started to write for him. From the middle of Book2, David Hilbert restart to write more and more by himself. Almost all notes of Book3 are written by himself. The reason is unknown.

B. Gordan's letter to Hilbert, 1888

The following is a letter from Gordan to Hilbert dating 26th December 1888. It is from the folder 116 of Hilbert Nachlasß, Niedersaechsische Staats- und Universitätsbibliothek Göttingen. Cod. Ms. 116 is the folder for letters from Gordan to Hilbert. The folder contains a letter dating December 22nd 1988 to thank Hilbert for the first of the three papers of Göttinger Nachrichten. Gordan congratulated Hilbert for the great work, but seems to have already had some anxity on the first theorem, i.e. the finite basis theorem. Some weeks later, Arthur Cayley, to whom Hilbert also sent his paper, replied on the same paper and his letters are also kept in Hilbert's Naclaß.

Transcription:

Erlangen 26 Dec. 88,

Geehrter Herr College!
Ihre Arbeit habe ich mit großen Interesse gelesen und gratulire Ihnen zu den daselbst gewonnenen Resultaten, welche von weittragender Bedeutung sind.

Wenn man einmal Ihren ersten Satz zugiebt, so sind die Folgerungen ziemlich einfach; aber in Bezug auf die Fassung dieses ersten Satzes habe ich Bedenken. Nicht als ob ich an seiner Nichtigkeit zweifelle, aber es scheint mir, daßSie vielleicht die Formen φ specialisiren können und dadurch die Entwicklung vereinfachen könnten.

Mit der Bitte, mich den Herren Lindemann und Hurwitz bestens zu empfehlen, verbleibe Ihnen nochmals gratulirend.
Ihr ganz ergebe⟨⟨ ner ⟩⟩
Gordan

[34]Kept as Cod. Ms. Hilbert 396.

Translation:

Erlangen 26 Dec. 88,

Dear Colleague!
I have read your work with a great interest and I congratulate you on the results obtained there, which are of long-range significance.

If one once admits your first theorem, then the conclusions are quite simple; but I have a reservation concerning the setting of this first theorem. Not as if I doubt its triviality, but it seems to me that you can perhaps specialize the forms φ and through it could simplify the development.

Asking you to send my best regards to Mr. Lindemann and Mr. Hurwitz, I remain again congratulating you.
Yours very devoted

Gordan

C. Original finite basis theorem

In this appendix, the original form of Hilbert's finite basis theorem will be elucidated. The original version of the finite basis theorem published in 1890 [31] is rather different from the version of the finite basis theorem now taught in the modern courses of algebra.

Let k be a field of numbers, i.e., a subfield of the complex numbers **C**. Hilbert's original formulation of the finite basis theorem was as follows:

> *for any sequence F_1, $F_2, \ldots$ of forms from $k[x_1, x_2, \ldots, x_n]$, there is a number m so that for any i there are forms A_1, $A_2, \ldots, A_m$ so that A_i equals to $A_1F_1 + \cdots + A_mF_m$.*

Modern formulation of the theorem is that any ideal in the ring of polynomials $k[x_1, x_2, \ldots, x_n]$ is finitely generated, where k is a field. In his 1890 paper, Hilbert proved a proposition (Satz), which is practically the modern version, by means of his original version. Then he applied the general proposition to the set of invariants to prove the first step of the proof of generalized Gordan theorem. The generalized proposition was unnamed, although the original version of the finite basis theorem was *named* as "Theorem I".

To prove the modern generalized version, one needs the law of the excluded middle and a weak version of axiom of choice called *axiom of dependent choice*. Thus, from the modern point of view, it is strange that Hilbert did not call the general proposition "Theorem I". However, from algebraic point of view, the general Gordan theorem for the case the field of rational numbers is essential and then we do not need to consider arbitary ideals, which might be uncountable set, but it is enough to consider countable ones as remarked Hilbert's 1897 lecture course on the algebraic invariant theory, c.f. [25].

To prove the theorem, Hilbert used the induction on the number of variables, i.e. n of $k[x_1, \ldots, x_n]$. For the base case $k[x_1]$, the forms F_1, $F_2, \ldots$ could be written as $a_1x_1^{r_1}$, $a_2x_1^{r_2}$, …. Take the smallest number v among the natural numbers r_1, r_2, …. Let m be a number so that $r_m = v$. Since we may assume that a_i is not zero, every $a_ix_1^{r_i}$ could be written as $(\frac{a_i}{a_m}x_1^{r_i-r_m})F_m$. This proves the theorem for the base case.

Note that there is no general way to compute the smallest number of a given non-empty set A of natural numbers. Assume that we know 8 belongs to A. If there is an algorithm to compute the truth value of $n \in A$, we simply check the truth values of the conditions $0 \in A$, $1 \in A$, $2 \in A$, …. The first true $n \in A$ represent the smallest element n. However, if there is no such an algorithm, we cannot compute the smallest element in general.

In the case of Hilbert's original finite basis theorem, A is the set of numbers r_i which appear in the infinite sequence $c_ix_1^{r_i}$. Hilbert described the sequence of forms as a non-stopping sequence (eine nicht abbrechende Reihe), but gave no other conditions. It could be even radamly defined. Then, of course, we cannot predict which r_i is smalled. Even if r_i is determined by a simple algorithm, the smallest r_m is not computable in general.

This could be easily seen by famous Brouwer style argument. Goldbach conjecture is that every even integer bigger than 5 could be represented by an addition of two primes. It has not been known if the conjecture is correct or not. Set c_i be constantly 1. Set r_i be 1, if the conjecture is correct for all even numbers greater than 5 and smaller than i. Set r_0 be 0 otherwise. If we know the smallest r_i, then we can check if it is 0 or 1. If it is 0, then conjecture is incorrect. If it is 1, then conjecture is correct.

After proving the base case, Hilbert proved the induction step by technical recursive computations.[35] Although it was highly complicated, the step involved only recursions and algebraic operations. Hence the induction step was computational. However, the proof was not computational at all, since the base case was not computable. Hilbert's "theology" was the *minimal number principle* that there is smallest r_m among for any non-stopping sequence of non-negative integers r_1, r_2, …. It was used only once in the proof, but it was propagated through the entire proof as it was used in the base case of a recursive proof.

Interestingly, Hilbert did not simply say "take the smallest v. He wrote it much more *operational* way. The following is an English translation of Hilbert's original argument in 1890 paper (p.144 of [1]). In it, the variable x is used instead of x_1 in our discussion.

> Let $c_1x^{r_1}$ be the first from in the given sequence whose coefficient c_1 is $\neq 0$. We now look for the next form in the sequence whose degree is $< r_1$; let this form be $c_2x^{r_2}$. We now look for the next form in the sequence whose degree is $< r_2$; let this form be $c_3x^{r_3}$. Continuing in this way, after at most r_1 steps, we com to a form F_m of the given sequence which is followed by no form of lower order. Since every

[35] Hilbert gave another proof for the two variables case, which is difficult to generalize to the general case. This proof uses another non-constructive argument, which is reducible to the minimum number principle.

form in the sequnece is divisible by this form F_m, m is the number with the property required by our theorem.